12. Exercise sheet | **Introduction to Order Theory**
Felsner/Micek | **Winter 2020/21**
Due for the exercise session: February 23., 2021

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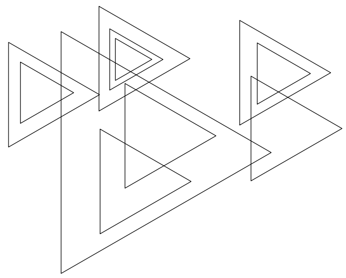
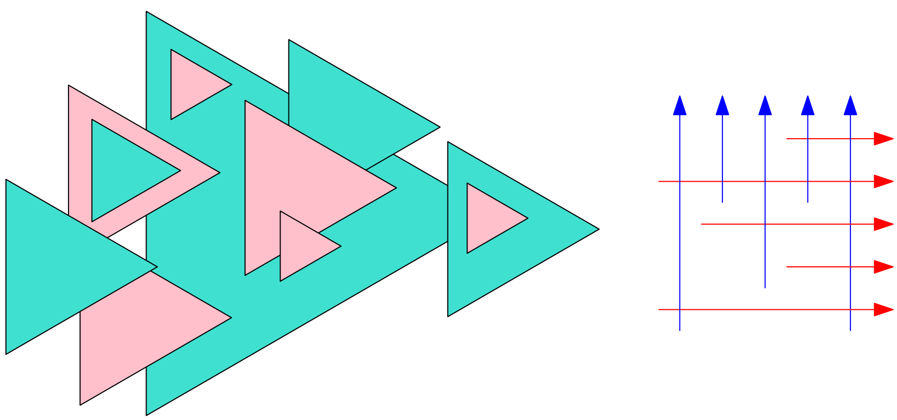
(1) What is the maximum dimension of a containment order of homothethic triangles in the plane?



(2) Let G be a bipartite graph with vertices represented by two-colored homothethic triangles in the plane and two triangles of different colors are adjacent if and only if they intersect. (Note that triangles of the same color can intersect but still they are not adjacent.) What is the maximum possible dimension of G seen as a poset of height 2 with one color being minimas and the other maximas?



(3) What is the maximum dimension of an intersection graph (seen as a height 2 poset again) of horizontal rays directed rightward and vertical rays directed upward (no two being colinear)?

(4) Prove that a poset of height 2 with a planar cover graph has dimension at most 4. *Hint*: Think of planar maps.

(5) Consider a quadrangulation G with opposite vertices a_1, a_2 on the outer face. Let $\mathcal{S}$ be a separating decomposition of G. For each vertex x in G, define two regions $R_1(x)$ and $R_2(x)$ as an analogue to regions of a Schnyder wood. Let $f_i(x)$ be the number of (bounded) faces of G in $R_i(x)$, for $i \in \{1, 2\}$. Show that $f_i(x)$ corresponds to the position of x on the equatorial line of $\mathcal{S}$.

(6) Let Q be a quadrangulation with opposite vertices a_1, a_2 on the outer face. A 2-orientation of Q is an orientation such that every vertex $\neq a_1, a_2$ has out-degree 2.

Show that there is a bijetion between separating decompositions and 2-orientations of Q.

(7) Show that $\dim(s', t'\,; n') \le \dim(s, t\,; n)$ for all $s \le s' < t' \le t < n' \le n$.

(8) Let P be an interval order and I be its interval representation (with no interval degenerated to a point). Show that for every linear extension L there is a marking function f of I such that for all x, y in P we have $m(x) < m(y)$ iff $x < y$ in L.

(9) Let $s + t \le n$ and $\lfloor t/s \rfloor + s - 1 \le k$. Show that $\dim(1, k\,; n) > t$.

The *fractional dimension* of P is the minimum $t \in \mathbb{R}$ such that there is a multi-realizer $\mathcal{R}$ of P such that $t = \max\big(\frac{|\mathcal{R}|}{|\mathcal{R}(a<b)|} : (a, b) \in \mathrm{inc}(P)\ \big)$. Here $\mathcal{R}(a < b) = \{L \in \mathcal{R} : a <_L b\}$, a multi-realizer is a multi-set realizer, and 'minimum t' can be replaced by 'infimum of all t' if you worry. Remark: It may be convenient to equip $\mathcal{R}$ with the uniform distribution, in this setting $1/t = \min\big(\mathrm{Prob}(a < b) : (a, b) \in \mathrm{inc}(P)\ \big)$.

(10) Determine the fractional dimension of $\mathcal{B}_n(1, t)$, i.e., of levels 1 and t in the Boolean lattice $\mathcal{B}_n$.

(11) Determine good upper bound for the fractional dimension of interval orders.

(12) Let $I(k, s)$ be the set of all interval orders with $k + s - 1$ elements, a connected comparability graph, and exactly k maximal antichains, each of them of size s. Let $a(k, s) = |I(k, s)|$. Determine the values of $a(16, 6)$ and $a(19, 7)$ and $a(30, 2)$.