11. Exercise sheet **Introduction to Order Theory**

Felsner/Micek **Winter 2020/21**

Due for the exercise session: February 16., 2021

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(1) Consider a symmetric chain decomposition $\mathcal{C}$ on the Boolean lattice $\mathcal{B}_n$. How many chains of size k are in $\mathcal{C}$ for $1 \leq k \leq n+1$?

(2) Show that, if $A_1, \dots A_m$ are distinct k-subsets of an n-set and $k \leq s \leq n-k$, then there exist distinct s-subsets $B_1, \dots, B_m$ such that $A_i \cap B_i = \emptyset$ for each $i = 1, ..., m$.

(3) Prove that the families $\binom{[n]}{\lfloor n/2 \rfloor}$ and $\binom{[n]}{\lceil n/2 \rceil}$ are the only antichains in $\mathcal{B}_n$ of size $\binom{n}{\lceil n/2 \rceil}$.

(4) [Littlewood Offord Problem] Let $a_1, \dots, a_n$ be a sequence of reals such that $|a_i| \geq 1$ for all $i \in [n]$. Let

$$P(a_1, \dots, a_n) = \{(\varepsilon_1, \dots, \varepsilon_n) \in \{-1, 1\}^n \mid -1 < \sum_{i=1}^{n} \varepsilon_i \cdot a_i < 1\}.$$

Show that $|P(a_1, \dots, a_n)| \leq \binom{n}{\lfloor n/2 \rfloor}$.

(5) Let $\mathcal{F}$ be a family of subsets of $[n]$ with no $A_1, \dots, A_{s+1} \in \mathcal{F}$ such that $A_1 \subsetneq A_2 \subsetneq \dots \subsetneq A_{s+1}$. Prove that the size of $\mathcal{F}$ is at most the sum of s largest binomials of the form $\binom{n}{i}$.

(6) Let $1 \leq s < r < n$ and let $\mathcal{F}$ be a family of r-subsets of $[n]$ such that for every $A \neq B \in \mathcal{F}$ we have $|A \cap B| \leq s$. Show that

$$|\mathcal{F}| \leq \frac{\binom{n}{s+1}}{\binom{r}{s+1}}.$$

(7) For each k with $1 \leq k \leq n/2$ find an intersecting family $\mathcal{F}_k$ of size 2^{n-1} in $\mathcal{B}_n$ such that the smallest set in $\mathcal{F}_k$ has size k.

(8) Fix $1 \leq k \leq n \leq 2k$ and show that if $\mathcal{F}$ is an antichain of subsets of $[n]$ each of size at least k, and not containing two sets whose union is $[n]$ then $|\mathcal{F}| \leq \binom{n-1}{k}$.

(9) A family of subsets $\mathcal{F}$ of $[n]$ is *distinguishing* if for every $x \neq y \in [n]$ there is $F \in \mathcal{F}$ so that $|F \cap \{x, y\}| = 1$. A family of subsets $\mathcal{F}$ of $[n]$ is *strongly distinguishing* if for every $x \neq y \in [n]$ there are $F_1, F_2 \in \mathcal{F}$ such that $x \in F_1 - F_2$ and $y \in F_2 - F_1$.

- What is the minimum size of a distinguishable subset of $[n]$?
- What is the minimum size of a strongly distinguishable subset of $[n]$?

(10) Let σ be a cyclic permutation of $[n]$ and let S be a family of k-arcs of σ with $\triangle_\sigma(S)$ we denote the σ-shadow of S, i.e., the set of all $(k-1)$-arcs contained in an arc of S. Show that unless $|S| = n$ we have $|\triangle_\sigma(S)| > |S|$.

(11) let $\mathcal{A}$ be a family of k-subsets of $[n]$ such that for every h tuple $(A_1, A_2, \dots, A_h)$ of sets from $\mathcal{A}$ we have $A_1 \cap A_2 \cap \dots \cap A_h \neq \emptyset$. Show that if $k \cdot h \leq (h-1)n$, then $|\mathcal{A}| \leq \binom{n-1}{k-1}$.

(12) Let $F_k(m)$ be the collection of the first m sets in the colex order on k-sets. Show that for $m \geq 1$ we have $|\triangle F_k(m+1)| \leq |\triangle F_k(m)| + (k-1)$.