9. Exercise sheet **Introduction to Order Theory**
Felsner/Micek **Winter 2020/21**
Due for the exercise session: January 26., 2021

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(1) Consider the following method of shuffling n cards. At each step, a card is chosen independently and uniformly at random and put on the top of the deck. We can think of the shuffling process as a Markov chain, where the state is the current order of the cards. Clearly, the chain is finite, irreducible, and aperiodic, so it has a unique stationary distribution.

a) Determine the stationary distribution.

b) Consider the following coupling (X_t, Y_t) of the described chain. Choose a position j uniformly at random from 1 to n and then obtain X_{t+1} from X_t by moving the jth card to the top. Denote the value of this card by C. To obtain Y_{t+1} from Y_t move the card with value C to the top. Argue that the coupling is valid and use it to show that the discussed chain is rapidly mixing.

c) Forget about the coupling and prove that at the moment that each card in the chain was picked and put at the top at least once the distribution of the current state is the stationary distribution.

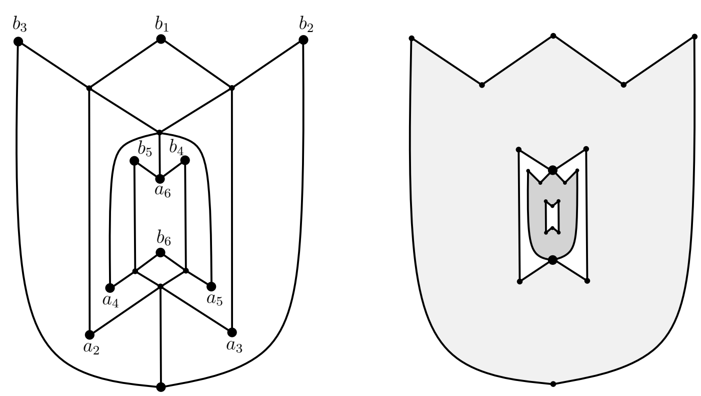
(2) Construct a family of posets $\{P_n\}$ with $|P_n| = n$ such that the Karzanov-Khachian chain has a mixing time $\Omega(n^3)$.

(3) Construct a family of posets $\{P_n\}$ with $|P_n| = n$ such that the Bubley-Dyer chain has a mixing time $\Omega(n^3)$.

(4) Show that for every $h \geq 1$ there is a poset P of height at most h with a planar diagram such that

$$\dim(P) \geq (4/3)h - 2.$$

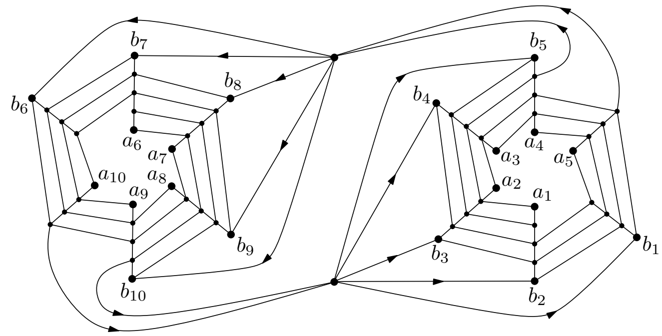
Hint: This figure:



(5) Show that for every $h \geq 1$ there is a poset P of height at most h with a planar cover graph such that

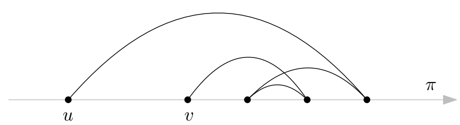
$$\dim(P) \geq 2h - 2.$$

Hint: The figure on the top of the other side:



(6) Let T be a tree. What is the maximum possible value of $\mathsf{wcol}_r(T)$? Give upper and lower bounds.

Let G be a graph and π be a linear order on $V(G)$. For $v \in V(G)$ and $r \geq 0$ we say that $u \in V(G)$ ist *r-strongly reachable* from v w.r.t. π in G if there is a path Q from v to u in G such that u is the minimum vertex of Q in π and for all internal vertices w of Q we have $w > v$ in π.



We define

$$\begin{aligned}\mathsf{SReach}^{\pi}_{r} &= \{u \in V(G) \mid u \text{ is } r\text{-strongly reachable from } v \text{ in } \pi\},\\ \mathsf{scol}_r(G) &= \min_{\pi} \max_{v \in V(G)} |\mathsf{SReach}^{\pi}_{r}(v)|.\end{aligned}$$

(7) Show that for every graph G and every integer $r \geq 0$ we have

$$\mathsf{scol}_r(G) \leq \mathsf{wcol}_r(G) \leq (\mathsf{scol}_r(G))^r\,.$$

(8) The *grid* $\boxplus_n$ of order n is the graph with the vertex set $\{(i,j) \mid i,j \in \{1,\dots,n\}\}$ whose two distinct vertices (i,j), (i',j') are adjacent iff $|i'-i|+|j'-j| = 1$. What are the possible asymptotics for functions f and g such that

$$f(r) \leq \mathsf{scol}_r(\boxplus_n) \leq \mathsf{wcol}_r(\boxplus_n) \leq g(r),$$

where n goes over all the natural numbers?

(9) Let G be an interval graph with $\omega(G) \leq k$. Prove that for every $r \geq 0$

$$\mathsf{wcol}_r(G) \leq \binom{r+k-1}{r}.$$

(10) Let G be a graph and p be an integer with $p \geq 1$. The *exact p-distance* graph $G^{[\#p]}$ is the graph with the same vertex set as G and two vertices are adjacent in $G^{[\#p]}$ if they are of distnace exactly p in G. Show that whenever p is odd we have

$$\chi(G^{[\#p]}) \leq \mathsf{wcol}_{2p-1}(G).$$

Hint: Consider an auxilliary greedy coloring along an ordering witnessing $\mathsf{wcol}_{2p-1}(G)$. To devise the final coloring consider balls $N^{\lfloor p/2 \rfloor}(v)$ in G, i.e. all vertices of distance at most $\lfloor p/2 \rfloor$ from v in G.