8. Exercise sheet | **Introduction to Order Theory**
Felsner/Micek | **Winter 2020/21**
Due for the exercise session: January 12., 2021

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(1) Let P be ranked with ranks of sizes $N_1, \dots, N_h$. We say that P has the LYM property, if $a_1, \dots, a_h$ are such that there is an antichain A in P which has a_i elements on rank i for $i = 1, \dots, h$, then

$$\sum_{i=1}^{h} \frac{a_i}{N_i} \leq 1.$$

Show that if P has the LYM property, then

$$\prod N_i! \leq e(P) \leq \prod N_i^{N_i}.$$

This little project goes from 2. to 4. with an application in 5. Let P_1, P_2, P_3 be orders on the same set X of elements, and let G_1, G_2, G_3 be their comparability graphs. We aim at showing that if $E(G_1) \cap E(G_2) \subseteq E(G_3)$, then $e(P_1) \cdot e(P_2) \geq e(P_3)$.

(2) Let $a_x = e(P_1 - x)/e(P_1)$ if $x \in \mathsf{Min}(P_3)$ and $a_x = 0$ otherwise. Show that $a \in \mathcal{C}(P_2)$.

(3) Let $b_x = e(P_2 - x)/e(P_2)$ for all x and show that $b \in \mathcal{A}(P_2)$.

(4) Use $a^\top b \leq 1$, induction, and the inequality $e(P_3) \leq \sum_{x \in \mathsf{Min}(P_3)} e(P_3 - x)$ to finally show $e(P_3) \leq e(P_1) \cdot e(P_2)$.

(5) Use the previous to show that for a 2-dimensional P with conjugate $\bar{P}$ we have

$$e(P) \cdot e(\bar{P}) \geq n!.$$

In this little project we aim at a probabilistic proof of the hook-formula for trees. Consider this algorithm:

random-max(T)
  $x \leftarrow$ uniform random vertex from T
  while $x \notin \mathsf{Max}(T)$
    $x \leftarrow$ uniform random vertex from $U(x)$
  return x

(6) Let $y \in \mathsf{Max}(T)$, show that

$$\mathsf{Prob}(y = \mathsf{random\text{-}max}(T)) = \frac{1}{n} \prod_{z<y} \frac{h_z}{h_z - 1}.$$

(7) Let $F(T) = \frac{n!}{\prod h_x}$. Use the previous exercise to show that $F(T) = \sum_{y \in \mathsf{Max}(T)} F(T-y)$ and conclude that $F(T) = e(T)$.