**7. Exercise sheet** **Introduction to Order Theory**

**Felsner/Micek** **Winter 2020/21**

Due for the exercise session: January 5., 2021

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(1) Identify the faces of the order polytope $\mathcal{O}(P)$, i.e., describe a class of combinatorial objects which are in bijection to the faces. *Hint*: It may be helpful to think of $\hat{P}$, i.e., of $P$ enriched with additional global **0** and **1** elements.

(2) What are the facets of the chain polytope $\mathcal{C}(P)$?

(3) For a graph $G = (V, E)$ we define

$$\mathcal{C}(G) = \{a \in [0,1]^V \mid \sum_{v \in C} a_v \leq 1 \quad \text{for every clique } C \text{ of } G\}.$$

If $P$ is an order with comparability graph $G$, then $\mathcal{C}(P) = \mathcal{C}(G)$.

Show that in general $\mathcal{C}(G)$ may have corners which are not characteristic vectors of stable sets.

(4) what is the maximum of $e(P)$ over all posets with $n$ elements and width at most $w$?

(5) what is the minimum of $e(P)$ over all posets with $n$ elements and height at most $h$?

(6) Consider the cover graph of randomly and uniformly selected $n$-element point set in the unit square. Let $X$ be the random variable counting number of edges in this graph. Evaluate $\mathsf{E}(X)$.

(7) For a poset $P$, let $m(P)$ be the maximum integer $m$ such that there are two disjoint subsets $A$ and $B$ of elements of $P$ with $|A| = |B| = m$ and $\forall_{a \in A,\ b \in B}\ a < b$ in $P$ or $\forall_{a \in A,\ b \in B}\ a \parallel b$ in $P$. Show that for every poset $P$ with $n$ elements and dimension $d$, we have

$$m(P) \geq \left\lfloor \frac{n}{2d} \right\rfloor.$$

*Hint:* Use the words low and high in your solution.

(8) Show that the maximum number of edges of a $K_{t,t}$-free incomparability graph of a 2-dimensional poset with $n$ elements is at most

$$2(t-1)n - \binom{2t-1}{2},$$

for every $t \geq 2$ and $n \geq 2t - 1$.
*Hint*: Show that such graphs are $(2t-2)$-degenerate, look at first $t$ points of a hypothetical counterexample.

(9) Show that every series-parallel order $P$ on $n$ elements satisfies

$$e(P) \cdot e(\bar{P}) = n!$$

As usual $\bar{P}$ is the conjugate of $P$.