6. Exercise sheet | **Introduction to Order Theory**
Felsner/Micek | **Winter 2020/21**
Due for the exercise session: December 15, 2020

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(1) Suppose that we are given a 3-colorable graph G but we do not know a witnessing coloring. Show how to color G with $\mathcal{O}(\sqrt{n})$ colors in polynomial time.

(2) A *circle order* is a containment order of discs in the plane. Show that standard examples are circle orders.

(3) Construct a family of arbitrarily large triangle-free graphs with the chromatic number in $\Omega(n^{1/3})$ where n is the number of vertices. Follow the description below.

Consider a finite projective plane of order q. So with q^2+q+1 points and q^2+q+1 lines. (You can browse the web for additional information on this object.) Fix an arbitrary ordering $\prec$ on points of the plane and an arbitrary ordering $\prec$ on lines of the plane Consider a graph G_q with vertices being all the point-line incidences. Two vertices (p_1, ℓ_1) and (p_2, ℓ_2) are adjacent in G_q if $p_1 \prec p_2$, $\ell_1 \prec \ell_2$ and (p_1, ℓ_2) is a point-line incidence (so is a vertex of G_q). Prove that

1. G_q is triangle-free;
2. if I is an independent set in G_q then reading I as the set of edges in the bipartite graph of point-line incidences, we have that I is acyclic.

Conclude the bound using a standard inequality $\chi \geq \frac{n}{\alpha}$ where α is the size of the largest independent set.

(4) Let P be an interval order and let D_P be its cover graph. Show that $\chi(D_p) \leq c\log(h(P))$. Try to optimize c.

(5) Prove that d-dimensional posets are containment orders of simple polygons with d-corners. Can you make the polygons convex?

(6) Let P be a poset and let $\mathcal{R}$ be a realizer. When we fix an ordering on linear extensions in $\mathcal{R}$ we say that $\mathcal{R} = (L_1, \ldots, L_k)$ is an *ordered realizer*. We say that two (incomparable) elements x, y of P *flip at coordinate* i $(1 \leq i < k)$ if $x < y$ in L_i and $y < x$ in L_{i+1}. The *crossing number* of an ordered realizer is the maximum over all pairs of elements of P of the number of times the pair flips. The *crossing number* of a poset is the minimum crossing number of its realizer. Clearly, d-dimensional orders have crossing number of at most $d-1$. Show that this bound is tight. Hint: Look at subposets of Boolean lattices.

(7) Show that the class $\mathcal{C}$ of disjointness graphs of families of grounded x-monotone curves satisfies $\chi \leq \binom{\omega+1}{2}$, i.e., it is χ-bounded. *Hint*: An graph G is a semi-comparability graph, if it admits an ordering $<$ such that it has no 4 vertices a, b, c, d with $a < b < c < d$ and $ab, bc, cd \in E(G)$, but $ac, bd \notin E(G)$. Verify that graphs in $\mathcal{C}$ are semi-comparability graphs and admit a partition into ω comparability graphs.

(8) Show that for every integer $k \geq 3$, every 3-dimensional order is a containment order of regular k-gons. (It is a deep result that there is a 3-dimensional order that is not a circle order.)