4. Exercise sheet **Introduction to Order Theory**

Felsner/Micek **Winter 2020/21**

Due for the exercise session: December 1, 2020

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(1) Let $<_0, <_1, \ldots, <_t$ be a set of linear orders of $[n]$ such that for $x, y, z \in [n]$ there is an i such that $y, z <_i x$. Define $S_{x,y} = \{i \ : \ 1 \leq i \leq t \text{ with } x <_i y\}$ and $A_x = \textsc{Max}\{S_{x,y} \ : \ x <_0 y\}$. Show that $A_x \neq A_y$ for all $x \neq y$.

(2) Show that $\dim(s', t'\,; n') \leq \dim(s, t\,; n)$ for all $s \leq s' < t' \leq t < n' \leq n$.

(3) Let $s + t \leq n$ and $\lfloor t/s \rfloor + s - 1 \leq k$. Show that $\dim(1, k\,; n) > t$.

The *fractional dimension* of P is the minimum $t \in \mathbb{R}$ such that there is a multi-realizer $\mathcal{R}$ of P such that $t = \max\left(\frac{|\mathcal{R}|}{|\mathcal{R}(a<b)|} : (a,b) \in \mathrm{inc}(P)\right)$. Here $\mathcal{R}(a < b) = \{L \in \mathcal{R} : a <_L b\}$, a multi-realizer is a multi-set realizer, and 'minimum t' can be replaced by 'infimum of all t' if you worry. Remark: It may be conveniant to equip $\mathcal{R}$ with the uniform distribution, in this setting $1/t = \min\left(\mathrm{Prob}(a < b) : (a,b) \in \mathrm{inc}(P)\right)$.

(4) Determine the fractional dimension of $\mathcal{B}_n(1,t)$, i.e., of levels 1 and t in the Boolean lattice $\mathcal{B}_n$.

(5) Determine good upper bounds for the fractional dimension of interval orders.

(6) Let $I(k,s)$ be the set of all interval orders with $k + s - 1$ elements, a connected comparability graph, and exactly k maximal antichains, each of them of size s.
Let $a(k,s) = |I(k,s)|$. Determine the values of $a(16,6)$ and $a(19,7)$ and $a(30,2)$.

(7) Let P be an interval order given with a representation. Show that for every linear extension L there is a marking function m such that $m(x) < m(y)$ iff $x < y$ in L.

(8) Let $N(m)$ be the minimal n such that the width of $\mathcal{B}_n$ is at least m. Let P be an interval order of width w. Show that

$$\dim(P) \leq N(w).$$

(9) Let P be an interval order which has no induced $\mathbf{t} + \mathbf{1}$. Show that

$$\dim(P) \leq N(t-1) + 1.$$

(10) Let P be an interval order of height h. Show that

$$\dim(P) \in O(\log\log(h)).$$