**3. Exercise sheet** **Introduction to Order Theory**
**Felsner/Micek** **Winter 2020/21**
Due for the exercise session: November 24, 2020

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(1) Let $U(x)$ and $D(x)$ be the *open* up-set/down-set of $x$ in $P$ that is

$$U(x) = \{y > x \mid y \in P\}, \qquad D(x) = \{y < x \mid y \in P\}.$$

Prove that for every interval order $P$, we have

$$|\{D(x) \mid x \in P\}| = |\{U(x) \mid x \in P\}|.$$

(2) Let $C_1, C_2$ be totally incomparable chains of $P$, i.e., for all $x \in C_1$ and $y \in C_2$ we have $x \| y$ in $P$. Show that

$$\dim(P) \leq 2 + \dim(P \setminus (C_1 \cup C_2)).$$

(3) Let $P$ and $Q$ be orders that both have a global minimum $\mathbf{0}$ and a global maximum $\mathbf{1}$ (and $\mathbf{0} \neq \mathbf{1}$ in both orders). Show that

$$\dim(P \times Q) = \dim(P) + \dim(Q).$$

(4) Let $D$ be a directed graph. An *arc-coloring* of $D$ is an assignment of colors to arcs such that consecutive arcs obtain different colors. *Arc chromatic number* of $D$, denoted by $A(D)$ is the least integer $k$ such that $D$ has an arc-coloring with $k$ colors. Show that

$$A(D) \geq \log \chi(D).$$

(5) Let $n$, $k$ be integers with $n, k \geq 2$. A *generalized shift graph* $G_n^k$ is the graph with the vertex set $V(G_n^k) = \binom{[n]}{k}$ an two $k$-tuples $\{x_1 < \cdots < x_k\}$, $\{y_1 < \cdots < y_k\}$ are adjacent if $x_2 = y_1, \ldots, x_k = y_{k-1}$ and $y_2 = x_1, \ldots, y_k = x_{k-1}$. Show that for every fixed $k$ the family of graphs $\{G_n^k\}_{n>1}$ has unbounded chromatic number

Hint: Use the previous exercise.

(6) **a.** What is the minimum length of a cycle in $G_n^{(k)}$?

**b.** What is the minimum length of an odd cycle in $G_n^{(k)}$?

(7) The *local chromatic number* of a graph $G$ is

$$\Psi(G) := \min_{c} \max_{v \in V(G)} |\{c(u) \mid u \in N(v)\}| + 1,$$

where the minimum is taken over all proper vertex-colorings $c$ of $G$.

Show that $\Psi\left(G_n^{(2)}\right)$ goes to infinity when $n$ grows.