2. Exercise sheet **Introduction to Order Theory**
Felsner/Micek **Winter 2020/21**
Due for the exercise session: November 17, 2020

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(1) Prove that the following conditions are equivalent:

a. G is a comparability graph of a poset of dimension at most 2;

b. G is a containment graph of intervals on a line;

c. G is a permutation graph;

d. G and its complement are both comparability graphs.

(2) Let $\dim^*(P)$ be the least integer d such that the elements of G can be embedded into $\mathbb{R}^n$ in such a way that for every x, y in P we have $x \leq y$ in P if and only if the point of x is less or equal the point of y in the product order on $\mathbb{R}^n$. Prove that $\dim(P) = \dim^*(P)$.

(3) Let $P = (X, \leq)$ be a poset. For a linear extension L of P, let $s(L)$ be a string over X with symbols aligned as elements in L. Prove that the set

$$\{s \mid s \text{ is a prefix of } s(L) \text{ for some linear extension } L \text{ of } P\}$$

is an antimatroid over X.

(4) Let P be a poset and x be an element of P. Let L be a linear extension of $P - x$. Show that one can always extend L to L^+ introducing x so that L^+ is a linear extension of P.

(5) Let P and Q be the posets and let $\dim(P) = d$. Show that

a. $\dim(P \setminus \{x\}) \in \{d-1, d\}$ for every $x \in P$,

b. $\dim(P \setminus \{x, y\}) \in \{d-1, d\}$ for every $x \in \min(P)$, $y \in \max(P)$, $x || y$,

c. $\dim(P + Q) \leq \max(\dim(P), \dim(Q), 2)$,

d. $\dim(P \times Q) \leq \dim(P) + \dim(Q)$.

(6) Let P be a poset and C be a chain in P. Prove that

$$\dim(P) \leq \dim(P - C) + 2.$$

(7) Let M be a subset of maximal elements of a poset P. Let $\mathrm{width}(P \setminus M) \leq w$. Show that

$$\dim(P) \leq w + 1.$$

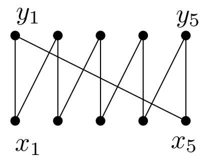
(8) How many antichains does the product $\mathbf{k} \times \mathbf{l}$ of two chains have?

Exercise (9) on the back.

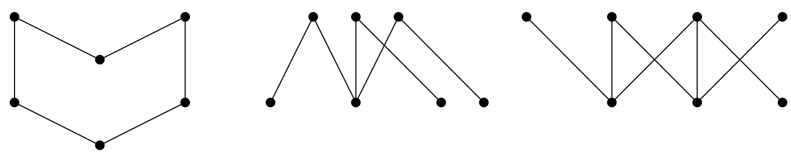
(9) A poset is 3-*irreducible* if it has dimension 3 and after removing any element the dimension drops to 2. There is a complete list of 3-irreducible posets (it includes some infinite families).

Prove that the dimension of the posets below is at least 3.

a. The *crown* C_n of order n is a poset on $2n$ elements $x_1, \ldots, x_n, y_1, \ldots, y_n$ with $x_i < y_i$ and $x_i < y_{i+1}$ for $i \in \{1, \ldots, n\}$ (cyclically) and no other strict comparabilities.



b. Three sporadic examples: the chevron, the spider, and one more.



c.