1. Exercise sheet **Introduction to Order Theory**
Felsner/Micek **Winter 2020/21**

Due for the exercise session: November 10, 2020

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(1) **a.** How many edges has the diagram of the Boolean lattice $\mathcal{B}_n$?

b. How many edges has the comparability graph of $\mathcal{B}_n$?

c. How many 3-chains does the Boolean lattice $\mathcal{B}_n$ have?

d. How many 2-antichains does the Boolean lattice $\mathcal{B}_n$ have?

(2) Given a collection of orders $(X, \leq_i)$ on a common ground-set X. Show that $(X, \bigcap_i \leq_i)$ is again a partial order.

(3) **a.** Let G be a comparability graph, show that G contains no odd cycle of length ≥ 5 as induced subgraph.

b. Let G be a incomparability graph, show that G contains no odd cycle of length ≥ 5 as induced subgraph.

(4) Let G be a bipartite graph, how many transitive orientations does G have?

(5) Let G be a graph with a girth $\mathsf{girth}(G)$. Show that if $\mathsf{girth}(G) > \chi(G)$, then G is a cover graph.

(6) Let P be an order. Characterize the pairs (x, y) with the property that

- $P + (x < y)$ is an order,
- $P - (x < y)$ is an order.

(7) For each m construct a partial order P_m with $\binom{m+1}{2}$ elements such that if $B_1, \ldots, B_k$ is a cover of P_m with the property that each B_i is a chain or an antichain, then $k \geq m$. (Later we will see that Greene-Kleitman Theory implies that every order with less than $\binom{m+1}{2}$ elements has such a cover with $k < m$.)

(8) Show that the following two conditions are equivalent

- P is a weak order,
- P is $(2 + 1)$-free.

(9) Let I be a family of n intervals on the real line. Show that either I contains $\lceil \sqrt{n} \rceil$ pairwise disjoint intervals or I contains $\lceil \sqrt{n} \rceil$ intervals sharing a common point.

(10) Let $P = (X, \leq)$ be an order with a weighting $w : X \to \mathbb{R}_+$ on the elements. Show that there is a weighting $g : \mathcal{A} \to \mathbb{R}_+$ of the antichains of P such that $\max(w(C) : C$ chain $) = \sum_A g(A)$ and $w(x) = \sum_{A: x \in A} g(A)$ for all $x \in X$.

(Note that this is a weighted version of the dual of Dilworth's Theorem.)