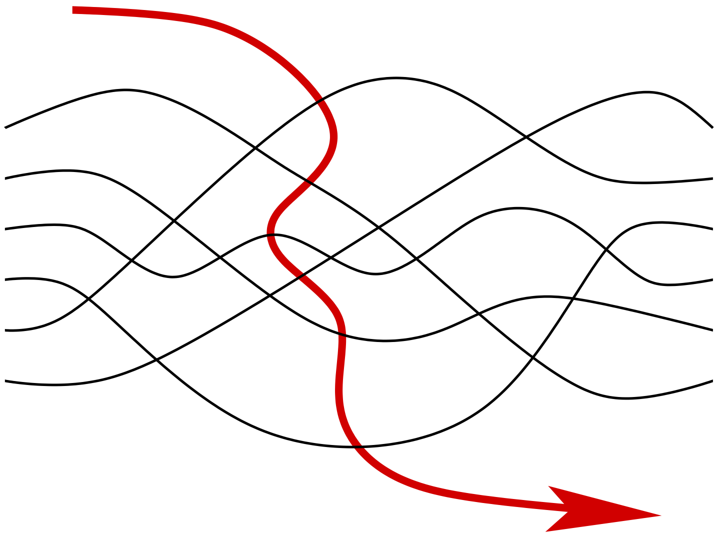
# Coding and Counting Arrangements of Pseudolines

August 31, 2010
Conf. on D&C Geometry
Bernoulli Center
EPFL



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Technische Universität Berlin

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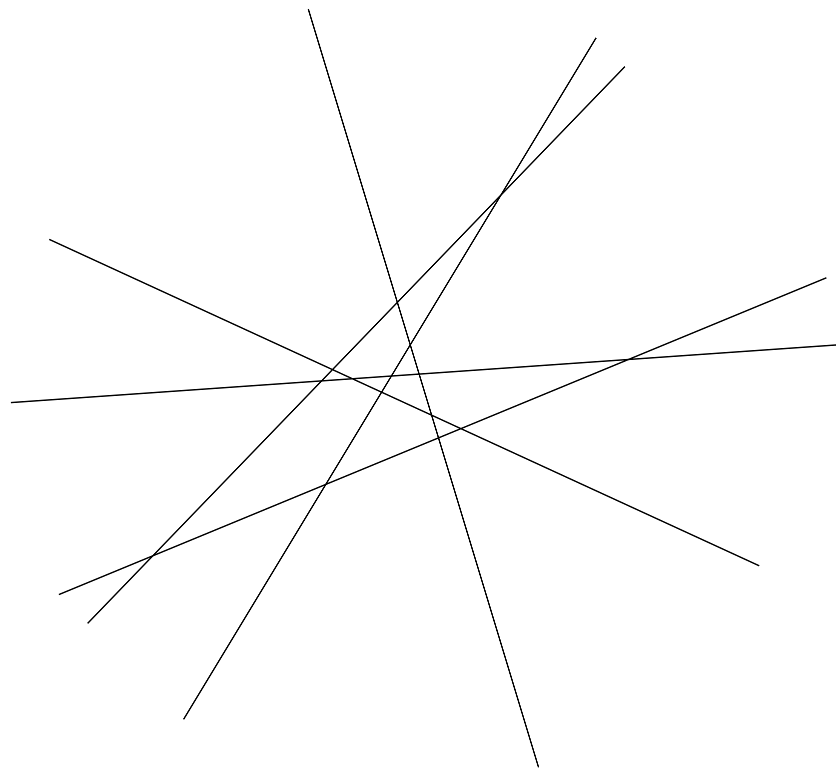
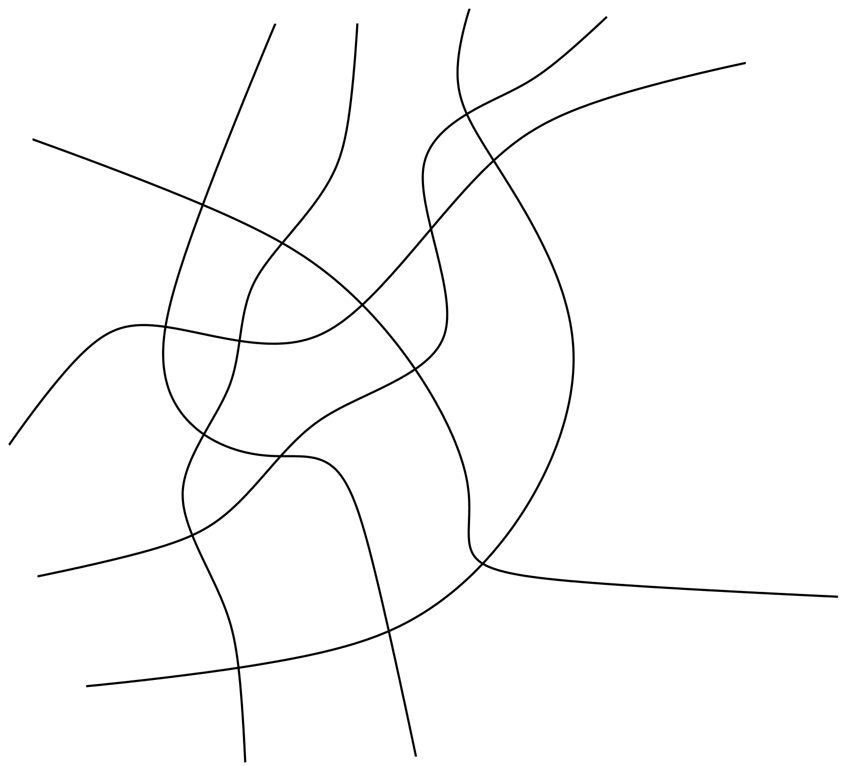
Charles University, Praha

# Arrangements of Lines



A (pairwise crossing) set of lines.

# Arrangements of Pseudolines



A 1-crossing set of curves extending to infinity on both sides.

# Our Version of Arrangements of Pseudolines

Euclidean: arrangements in $\mathbb{R}^2$ and not in $\mathbf{P}$.

simple: no multiple crossings.

marked: a special unbounded cell is the north-cell.

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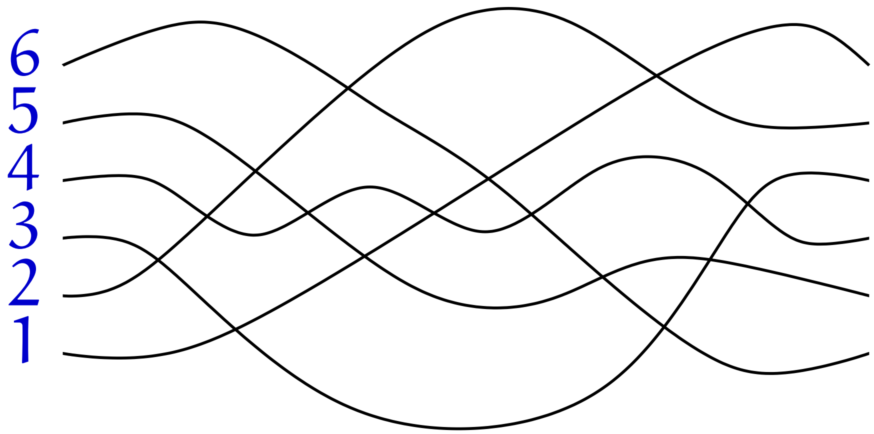
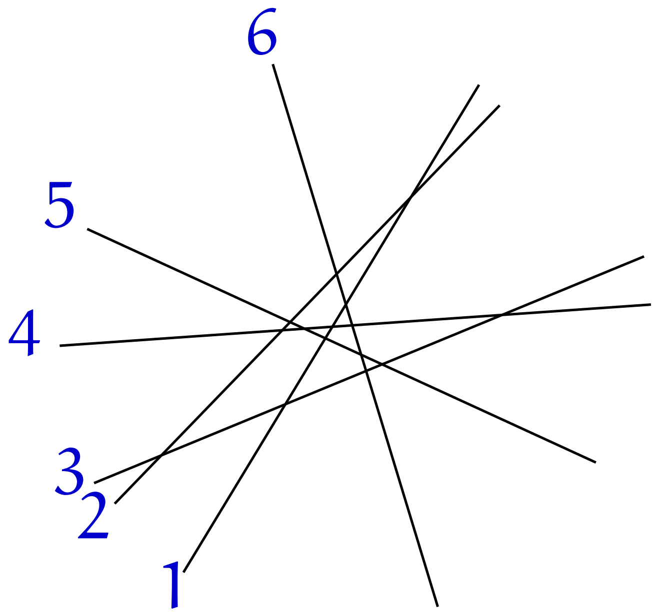
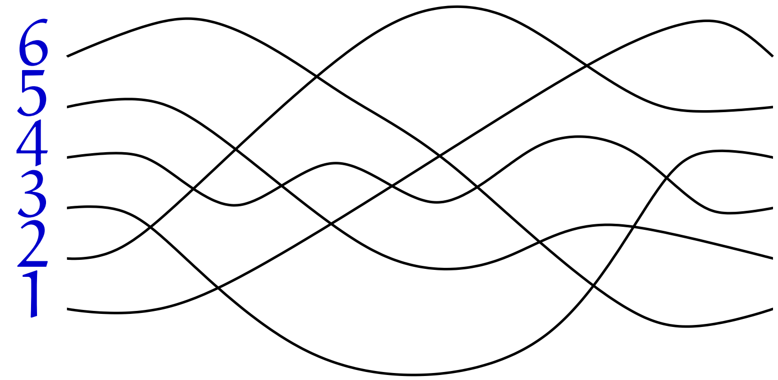



A 1-crossing set of $x$-monotone curves extending to infinity on both sides.

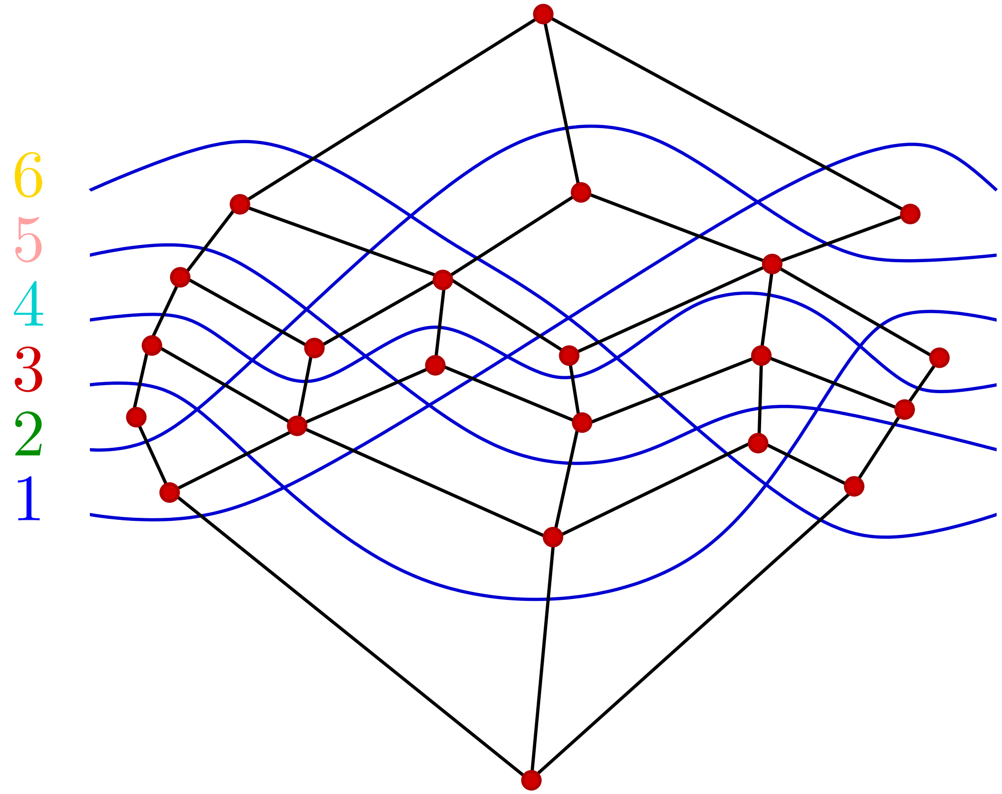
# Isomorphism




The two arrangements are isomorphic.

# Dual of an Arrangement



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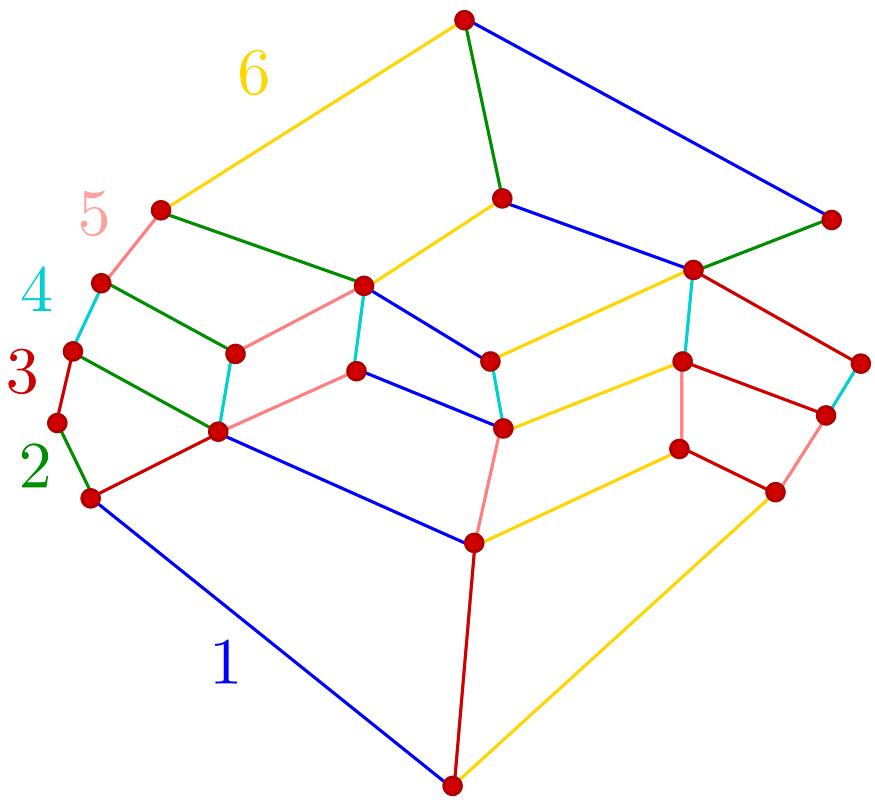
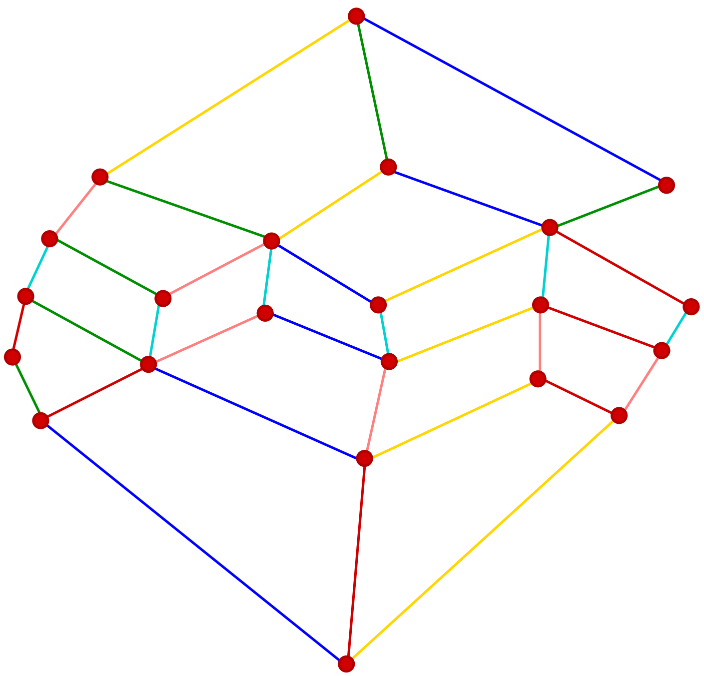
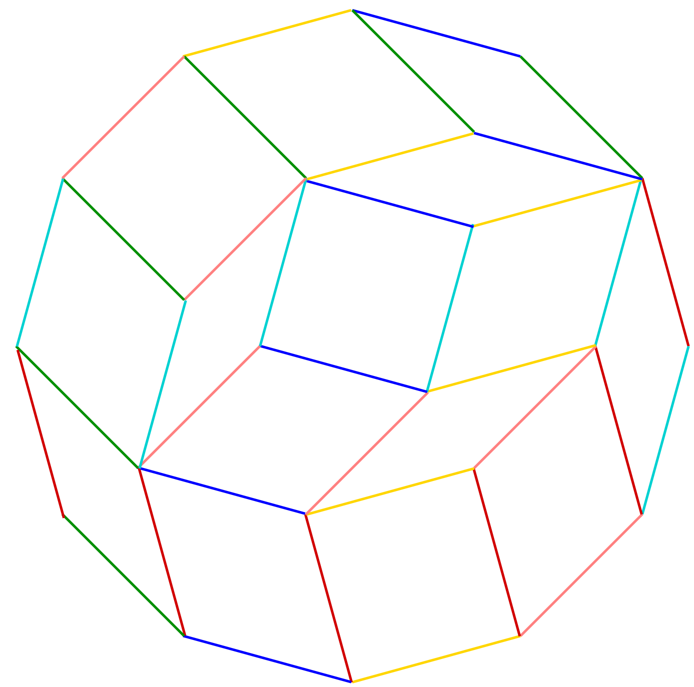
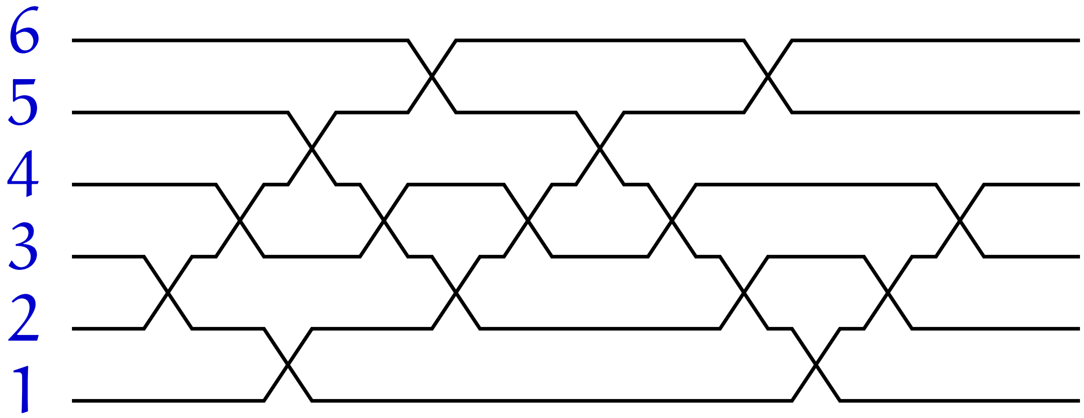


# Zonotopal Tiling

A tiling of an $2n$-gon with rhombic tiles.

# Wiring Diagram




Confine the $n$ pseudolines to $n$ horizontal wires and add crossings as Xs. (Goodman 1980)

# Counting Arrangements

$B_n$ number of isomorphism classes of simple arrangements of $n$ pseudolines. It is known that $B_n \approx 2^{bn^2}$ We are interested in the value of $b$.

**Upper bound**

- Knuth 92: $B_n \leq 3^{\binom{n}{2}} \implies b \leq 0,7924$.
- Felsner 97: $b \leq 0,6974$.
- New: $b \leq 0,6571$.

# Counting Arrangements

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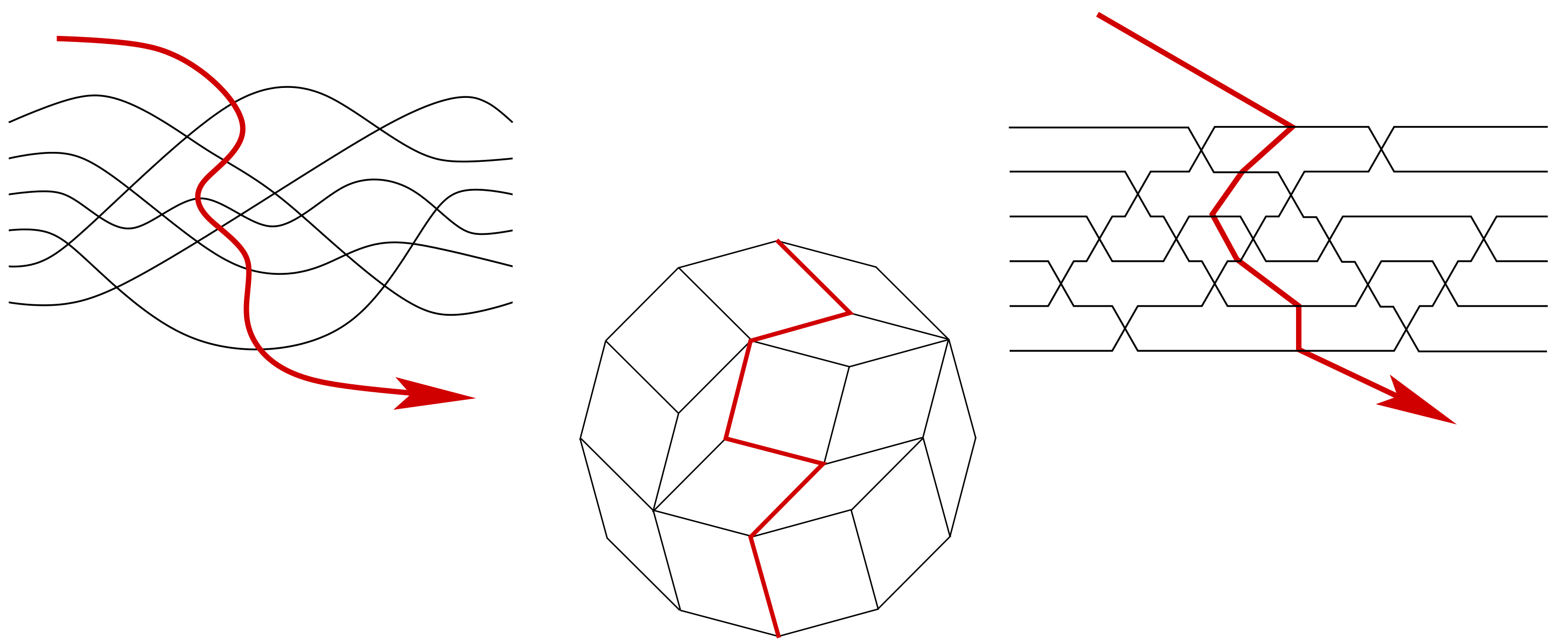
**Upper bound**: New: $b \leq 0,6571$.

**Lower bound**

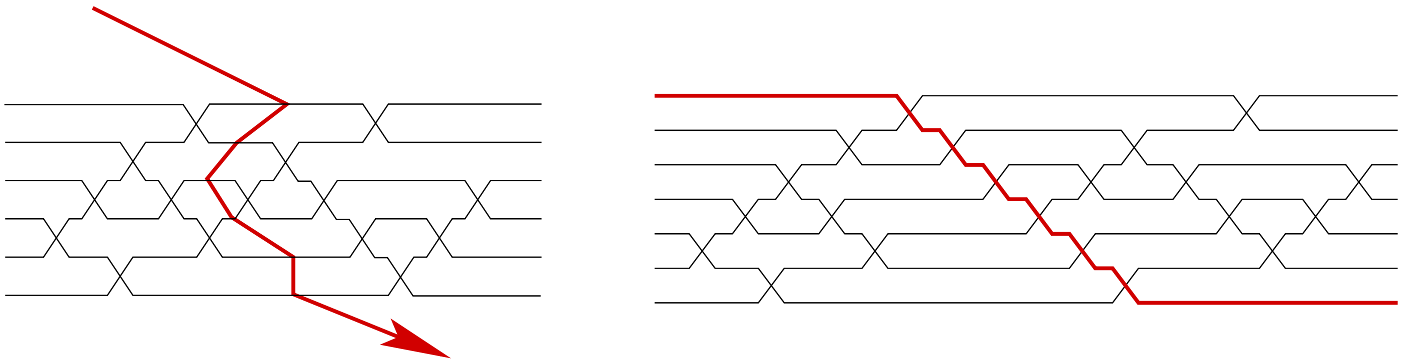
- Goodman and Pollack 84: $b \geq 0,1111$.
- Knuth 92: $b \geq 0,1666$.
- New: $b \geq 0,1888$.

# Cut-Paths

A curve from the north-cell to the south-cell crossing each pseudoline in a single edge.



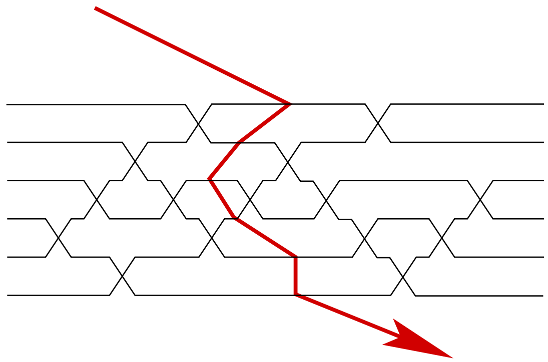
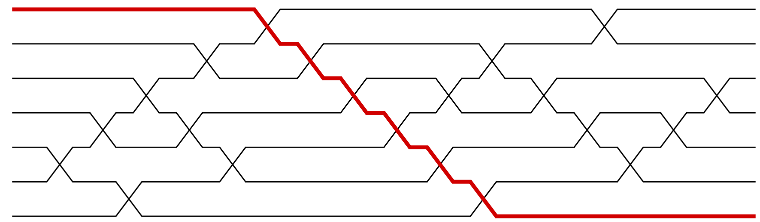
# Cut-Paths



If $\gamma_n$ is the maximal number of cut-paths of an arrangement of $n$ pseudolines, then

$$B_n \leq \gamma_{n-1} \cdot B_{n-1} \leq \gamma_{n-1} \cdot \gamma_{n-2} \cdot \ldots \cdot \gamma_2 \cdot \gamma_1.$$

# Cut-Paths

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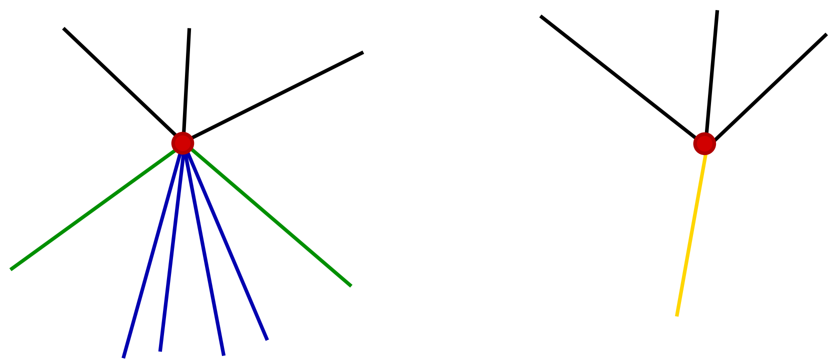
Task: Find good bounds on $\gamma_n$.

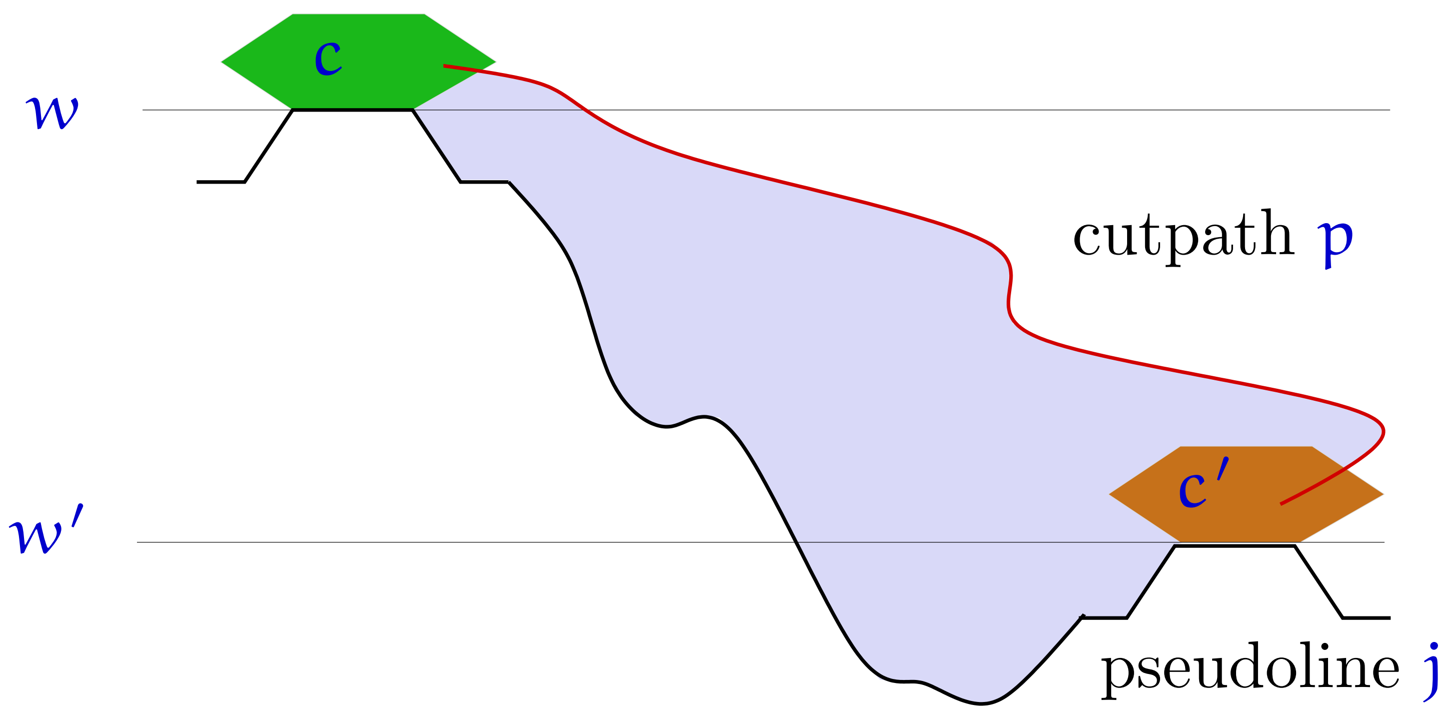
# Edges of a Cut-Paths

- We distinguish left, middle, right and unique edges on a cut-paths



# The Key Lemma

**Lemma.** [Knuth] For every pseudoline j and every cutpath p: p sees a middle of color j at most once.



# Encoding Cut-Paths I

With a cutpath $p$ we associate two combinatorial objects:

- A set $M_p \subset [n]$ consisting of all $j$ such that pseudoline $j$ is crossed by $p$ as a middle.
- A binary vector $\beta_p = (b_0, b_1, \ldots, b_{n-1})$ such that $b_i = 1 \iff p$ takes a right when crossing wire $i$.

**Fact.** The mapping $p \to (M_p, \beta_p)$ is injective.

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**Fact.** The mapping $p \to (M_p, \beta_p)$ is injective.

$$\gamma_n \leq 2^n \, 2^n = 4^n.$$

# Encoding Cut-Paths II

If $|M_p| = k$, then we only need $n - k$ entries of $\beta_p$.

Redefine $\beta_p$ so that $b_i$ encodes the left/right step at the $i$th lookup.

$$\gamma_n \leq \sum_{k=0}^{n} \binom{n}{k} 2^{n-k} = 2^n \, (1 + \frac{1}{2})^n = 3^n.$$

# Reversed Cut-Paths

We don't need an entry of $\beta_p$ when taking a unique.

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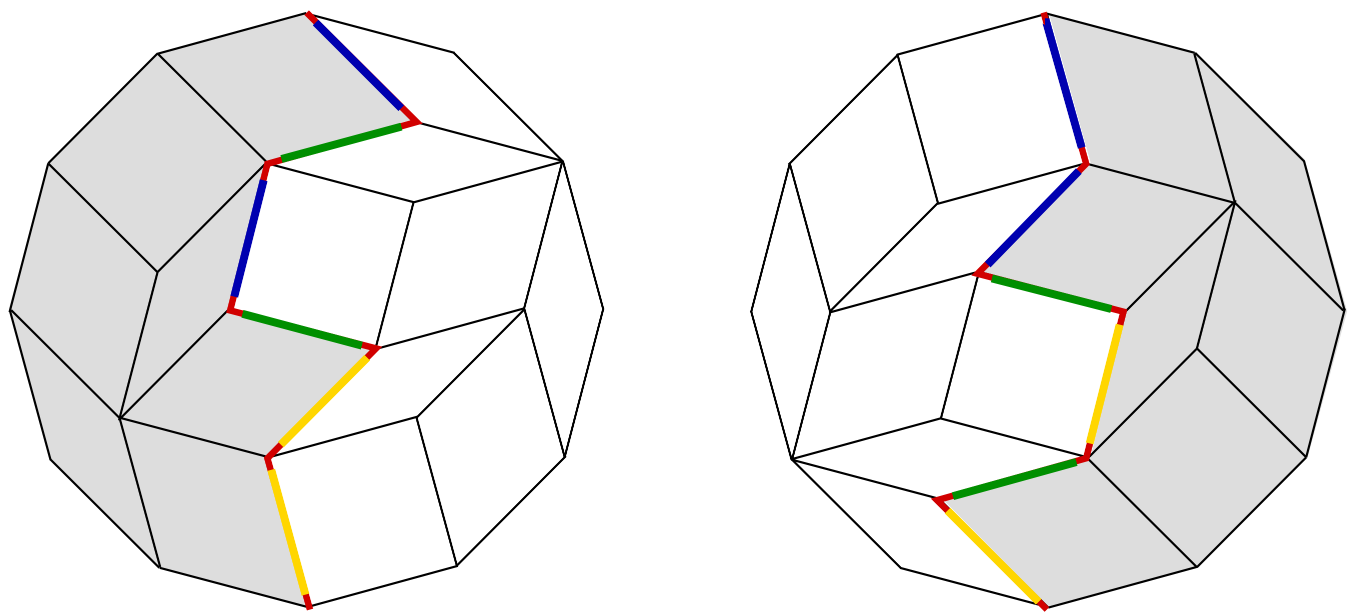
**Lemma.** A middle of $p$ is a unique of the reversed cut path.



# Encoding Cut-Paths III

If $\Gamma(k, r)$ is the number of cutpaths that take $k$ middles and $r$ unique edges, then $\Gamma(k, r) \le \binom{n}{k} 2^{n-k-r}$ and by the reversal symmetry $\Gamma(k, r) \le \binom{n}{r} 2^{n-k-r}$.

**Lemma.** $\Gamma(k, r) \le \min\left\{\binom{n}{k}, \binom{n}{r}\right\} 2^{n-k-r}$.

# Encoding Cut-Paths III

$$\begin{aligned}
\gamma_n &\le \sum_{k,r} \Gamma(k,r) \le \sum_{k,r} \min\left\{\binom{n}{k},\binom{n}{r}\right\} 2^{n-k-r} \\
&\le 2 \cdot 2^n \sum_{k=0}^{n} \binom{n}{k} 2^{-k} \sum_{r \ge k} 2^{-r} \\
&= 2^{n+1} \sum_{k=0}^{n} \binom{n}{k} 2^{-2k} \sum_{j \ge 0} 2^{-j} \\
&= 2^{n+2} \left(1 + \frac{1}{4}\right)^n = 4 \left(\frac{5}{2}\right)^n
\end{aligned}$$

**Corollary.** $\log_2(B_n) \le 0.6609 n^2$ for $n$ large.

# The Last Improvement

We have a slightly improved bound on $\Gamma(k, r)$.

**Definition.** A $k$-transversal of a partition $\Pi = (B_1, \ldots, B_h)$ of $[n]$ is a $k$-element subset $A$ of $[n]$ such that $|A \cap B_i| \leq 1$ for each $i \in \{1, \ldots, h\}$.

For $n \geq h \geq k$, let $P(n, h, k)$ be the maximum number of $k$-transversals a partition $\Pi = \{B_1, \ldots, B_h\}$ of $[n]$ with $h$ blocks can have.

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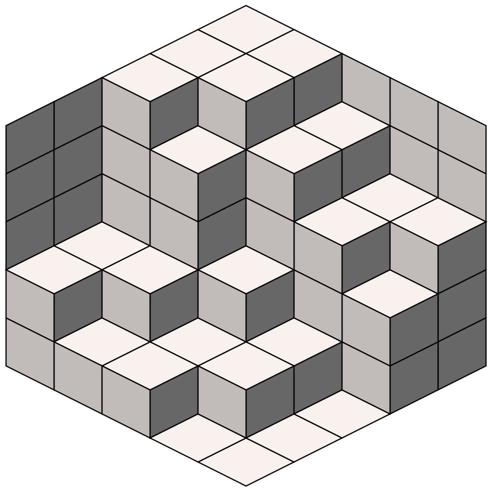
For $n \geq h \geq k$, let $P(n, h, k)$ be the maximum number of $k$-transversals a partition $\Pi = \{B_1, \ldots, B_h\}$ of $[n]$ with $h$ blocks can have.

We show $\Gamma(k, r) \leq 2^{n-k-r} P(n, n-r, k) \leq 2^{n-k-r} \binom{n-r}{k} \left(\frac{n}{r}\right)^k$.

This yields an improvement from $0.6609$ to $0.6571$.

# The Lower Bound



The MacMahon formula for the number of plane partitions in $n \times n \times n$, i.e., rhombic tilings of a hexagon with all sides of length $n$ is

$$\mathbf{P}(n) = \prod_{a=0}^{n-1} \prod_{b=0}^{n-1} \prod_{c=0}^{n-1} \frac{a+b+c+2}{a+b+c+1}.$$

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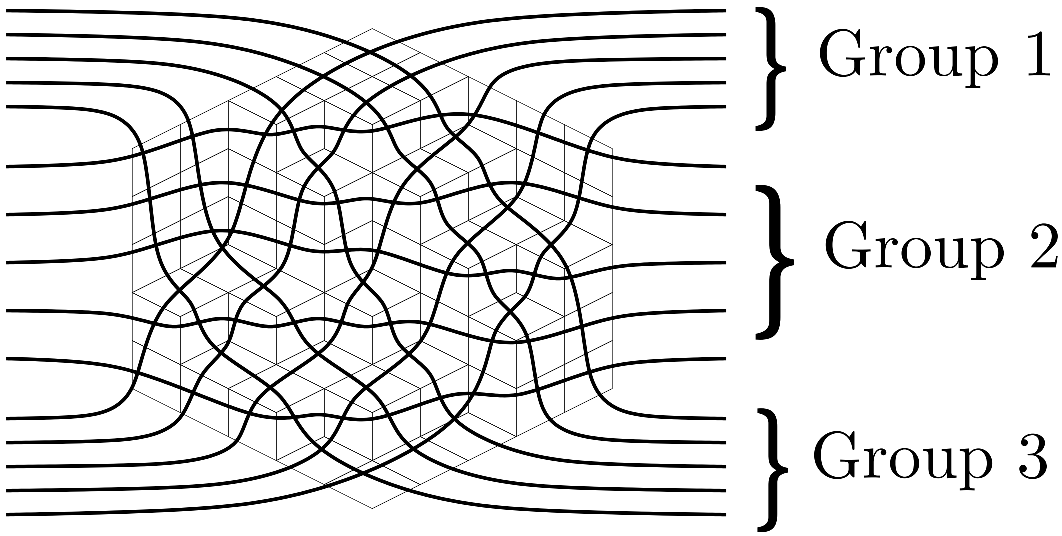


The construction implies $B_{3n} \geq \mathbf{P}(n)\, {B_n}^3$.

# The Lower Bound

The rest is a Maple supported computation:

$$\ln \prod_{a=0}^{n-1} \prod_{b=0}^{n-1} (a+b+k+1) \approx \int_{x=0}^{n} \int_{y=0}^{n} \ln(x+y+k+1)\, dy\, dx$$

yields

$$\ln \mathbf{P}(n) \approx \left(\frac{9}{2}\ln(3) - 2\ln(2)\right) n^2$$

and finally:

**Theorem.** The number $B_n$ of arrangements of $n$ pseudo-lines is at least $2^{0.1887\, n^2}$.

# Conclusion

There is a huge gap between 0.188 and 0.657.

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Thank you.