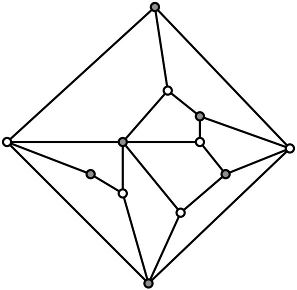
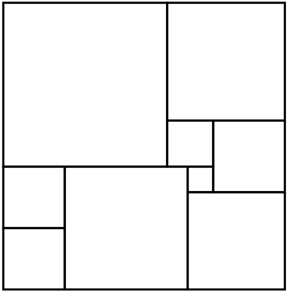
## More Representation Problems

Add some conditions.

- Find a representing rectangulation $R$ in a square such that all rectangles intersect the diagonal.
  (We saw a solution)
- Find a representing rectangulation $R$ such that all inner rectangles are squares.
  *The dissection of rectangles into squares*
  Brooks, Smith, Stone and Tutte 1940.

# Segment Contact Squarings



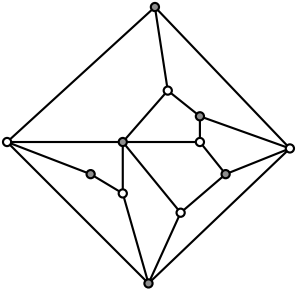


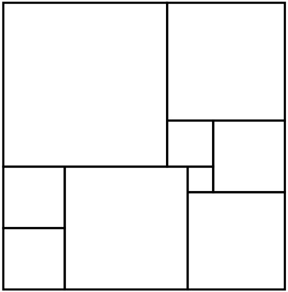
# Squarings a la BSST

View the bipolar graph as electrical network with edges of resistance 1 Ohm. Consider an $s \to t$ flow in this network. The distribution of current in edges corresponds to a squaring.
Based on this theory they give explicit solutions:

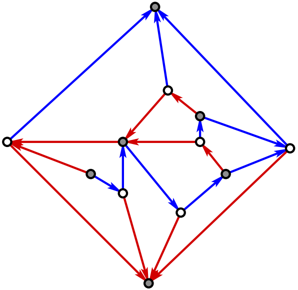
$$\mathrm{size}(i,j) = \# \text{ spanning trees } T \text{ with } (i,j) \text{ on the } s \to t \text{ path in } T - \# \text{ spanning trees } T \text{ with } (j,i) \text{ on the } s \to t \text{ path in } T.$$

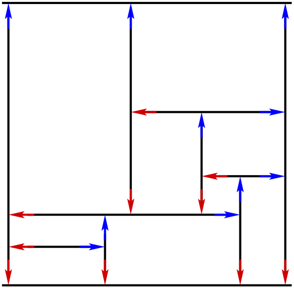
# Segment Contact Squarings





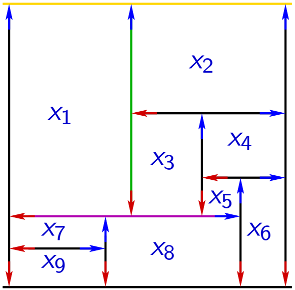
# Computing a Segment Contact Squaring





Step I: Compute a separating decomposition on $Q$.
This yields a rectangular dissection.

# Computing a Segment Contact Squaring



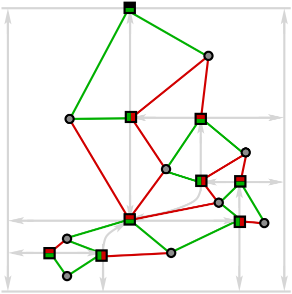

- $x_1 = x_2 + x_3$
- $x_1 + x_3 + x_5 = x_7 + x_8$
- $x_1 + x_2 = 1$

Step II: Set up a linear system of equations: $A_S \cdot x = e_1$

- $A_S$ is a square matrix.

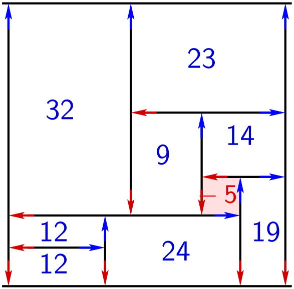
**Theorem.** $\det(A_S) \neq 0$.

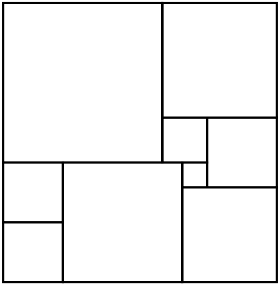
# Computing a Segment Contact Squaring



- $A_S$ is a bipartite adjacency matrix of a bipartite graph $H$.
- $\det(A_S) = \sum_\pi sign(\pi) \; \prod a_{i,\pi(i)}$
- $\det(A_S) = \sum_M sign(M)$, with $M$ perfect matching of $H$.
- $sign(M) = sign(M')$ for all $M$ and $M'$ perfect matching.
- $H$ has a perfect matching.

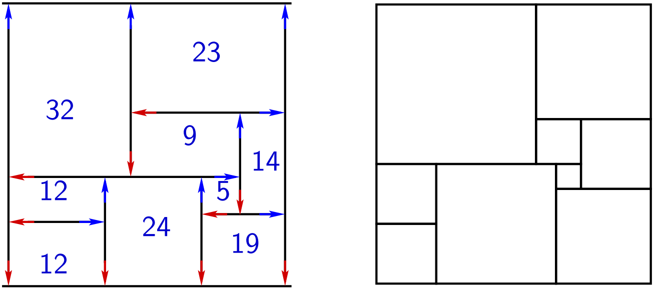
# Squarings with Segment Contacts



Step III: Flip negative faces to get the good separating decomposition and the squaring.

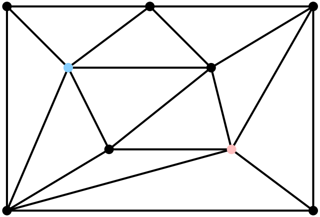
# Squarings with Segment Contacts

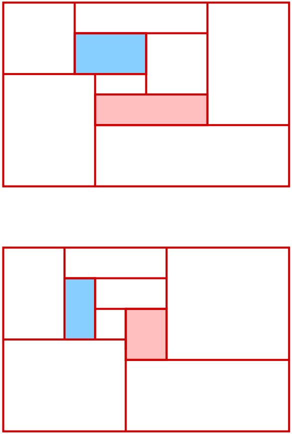



Step III: Flip negative faces to get the good separating decomposition and the squaring.

# Rectangular and square duals

# Rectangular Duals

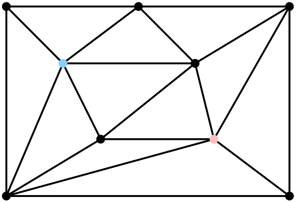


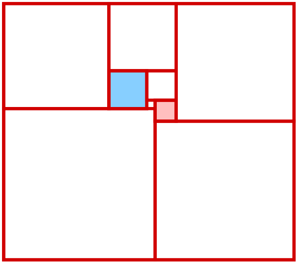


Rectangular duals are not unique.
They exist if the triangulation is 4-connected.

# Squarings for Inner Triangulations

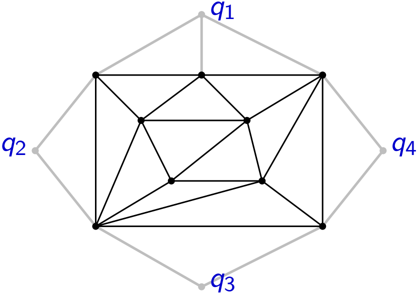
The squaring is unique.





O. Schramm *Square Tilings with prescribed Combinatorics.* 1993

# A Polyhedral view on Squarings



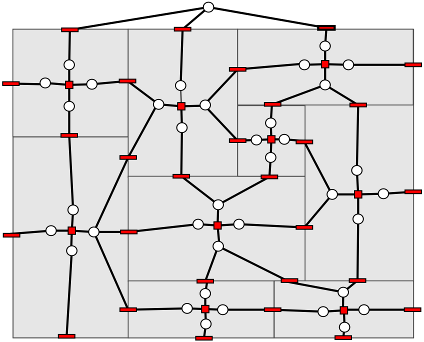

$P = \{x \in \mathbb{R}^V_{\geq 0} : \sum_{i \in \gamma} x_i \geq 1 \quad \text{for all } q_1 \to q_3 \text{ paths } \gamma\}$

solution vector for $\min(\sum_i x_i^2 : x \in P)$ yields squaring.

L. Lovász *Geometric Representations of Graphs*, 2009, Sec. 6.3.2.

# Computing Squarings with Systems of Equations

transversal structure $\Longrightarrow$ rectangular dissection.

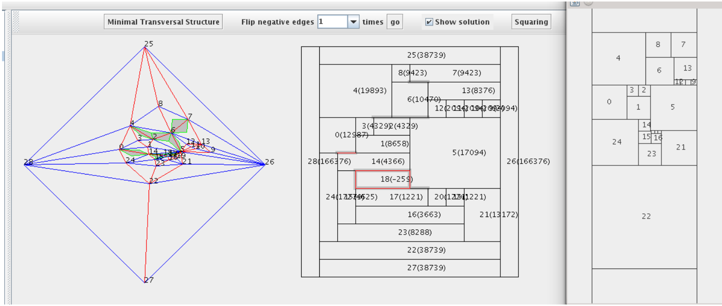


- red rectangles $=$ variables
- white circles $=$ equations

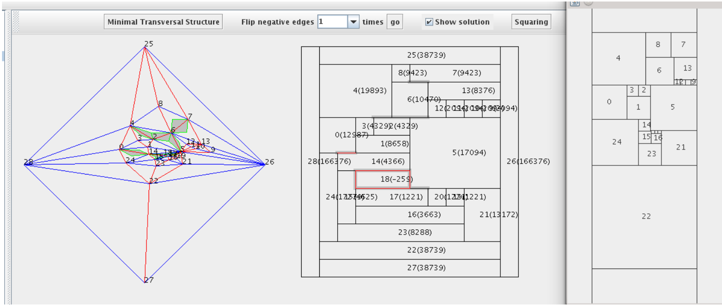
# The algorithm

1. $T$ transversal structure on a 4-triangulation $G$.
2. Calculate the solution $x_T$ of the equation system $Ax = b$.
3. If $x_T \geq 0$
   - Construct a square contact representation from $x_T$.
4. Else
   - $T'$ obtained by 'reverting bad-directed cycle' in $T$.
   - Set $T := T'$ and go to 2.

# The program


Minimal Transversal Structure
Flip negative edges
times
go
Show solution
Squaring

# The program




In the lecture we saw a demo.