Outline

Distributive Lattices and Markov Chains

Coupling from the Past

Mixing time on α-orientations

A General problem: Sampling

- Ω a (large) finite set
- $\mu : \Omega \to [0, 1]$ a probability distribution, e.g. uniform distr.

Problem. Sample from Ω according to μ.

i.e., $\Pr(\text{output is } \omega) = \mu(\omega)$.

There are many hard instances of the sampling problem.
Relaxation: *Approximate sampling*

i.e., $\Pr(\text{output is } \omega) = \widetilde{\mu}(\omega)$ for some $\widetilde{\mu} \approx \mu$.

Applications of (approximate) sampling:

- Get hand on typical examples from Ω.
- Approximate counting.

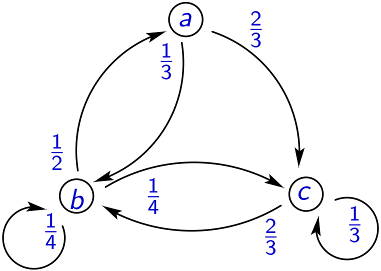
Preliminaries on Markov Chains

M transition matrix

- format $\Omega \times \Omega$
- entries $\in [0,1]$
- row sums $= 1$ (stochastic)

Intuition:

$$\mathbf{M} = \begin{bmatrix} 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{2}{3} & \frac{1}{3} \end{bmatrix}$$



M specifies a random walk

Ergodic Markov Chains

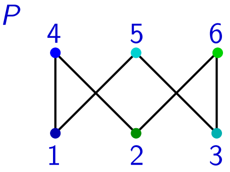
$\mathbf{M}$ is ergodic (i.e., irreducible and aperiodic)
$\implies$ multiplicity of eigenvalue 1 is one
$\implies$ unique π with $\pi = \pi\mathbf{M}$.

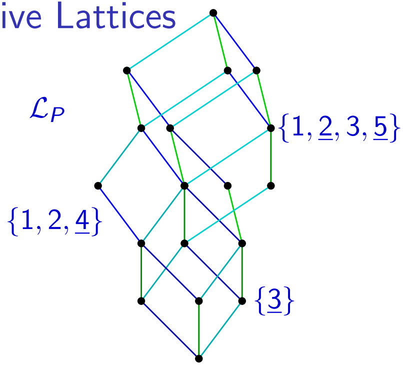
Fundamental Theorem.

$$\mathbf{M} \text{ ergoic} \implies \lim_{t\to\infty} \mu_0 \mathbf{M}^t = \pi.$$

$\mathbf{M}$ symmetric and ergodic
$\implies$ $\mathbf{M}^T \mathbb{1}^T = \mathbf{M}\mathbb{1}^T = \mathbb{1}^T$, hence $\mathbb{1}\mathbf{M} = \mathbb{1}$
$\implies$ π is the uniform distribution.

Markov Chains for Distributive Lattices





Lattice Walk

(A *natural* Markov chain on $\mathcal{L}_P$)

Identify state with downset D

- choose $x \in P$ & choose $s \in \{\uparrow, \downarrow\}$
- depending on s move to $D + x$ or $D - x$ (if possible)

Fact. The chain is ergodic and symmetric, i.e, π is uniform.

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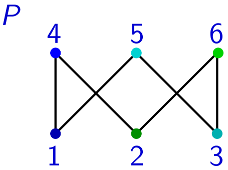

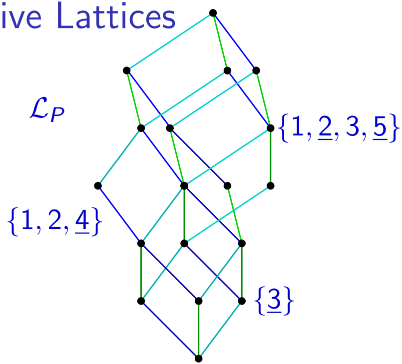

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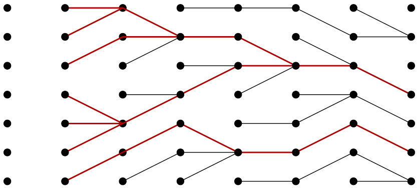
Distributive Lattices and Coupling From the Past



Theorem. The state returned by Coupling-FTP is exactly(!) in the stationary distribution.

The lattice walk on distributive lattices has the property:

- $x <_\Omega x' \implies f(x) <_\Omega f(x')$.

Theorem. On distributive lattices Coupling-FTP only requires the observation of two elements. observe The chain is ergodic and symmetric, i.e, π is uniform.

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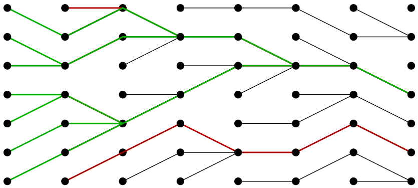

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Mixing Time

$\mu_x^t = \delta_x \mathbf{M}^t$ the distrib. after t steps when start is in x

$\Delta(t) := \max(\|\mu_x^t - \pi\|_{VD} \ : \ x \in \Omega)$

$\tau(\varepsilon) = \min(t \ : \ \Delta(t) \leq \varepsilon)$

- $\tau(\varepsilon)$ is the mixing time.
- $\mathbf{M}$ is rapidly mixing $\iff$ $\tau(\varepsilon)$ is a polynomial function of $\log(\varepsilon^{-1})$ and the *problem size*.

Big Challenge. Find interesting rapidly mixing Markov chains

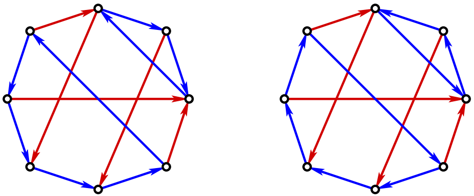
Example.

- Matchings (Jerrum & Sincair '88)
- Linear Extensions (Karzanov & Khachiyan '91 / Bubley & Dyer '99)
- Planar Lattice Structures, e.g. Dimer Tilings (Luby et al. '93)

Lattices of α-Orientations

Definition. Given $G = (V, E)$ and $\alpha : V \to \mathbb{N}$.
An α-orientation of G is an orientation with
$\mathrm{outdeg}(v) = \alpha(v)$ for all v.

Example.

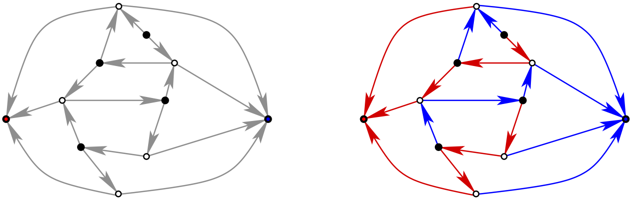


Two orientations for the same α.

Example: 2-Orientations

G a planar quadrangulation, let

- $\alpha(v) = 2$ for each inner vertex and $\alpha(v) = 0$ for each outer vertex.



A bijection 2-orientations $\longleftrightarrow$ separating decompositions

Counting and Sampling

Counting α-orientations is #P-complete for

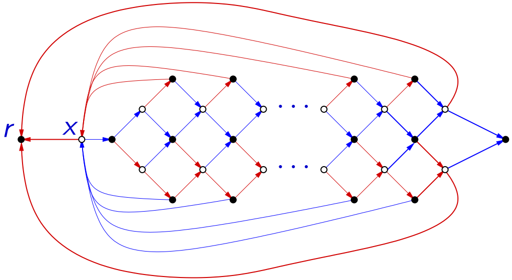
- planar maps with $d(v) = 4$ and $\alpha(v) \in \{1, 2, 3\}$ and
- planar maps with $d(v) \in \{3, 4, 5\}$ and $\alpha(v) = 2$.

Approximate Counting

Fact. The fully polynomial randomized approximation scheme for counting perfect matchings of bipartite graphs (Jerrum, Sinclair, and Vigoda 2001) can be used for approximate counting of α-orientations.

- What about the lattice walk?

Bad news



Theorem. Let Q_n be the quadrangulation on $5n+1$ vertices shown in the figure. The lattice walk on 2-orientations of Q_n has $\tau(1/4) > 3^{n-3}$.

- $|\Omega_c| = 1$
- $|\Omega_L| = |\Omega_R| \geq \frac{1}{2}(3^{n-1} - 1)$.

The lattice has "hour-glass" shape.

A Positive Result

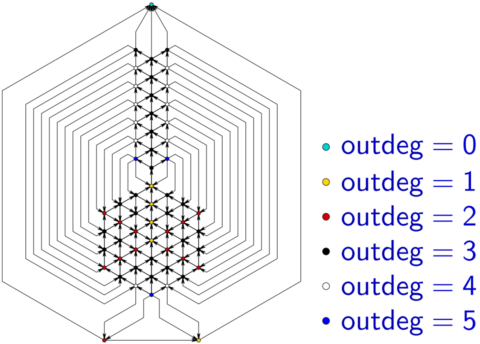
Theorem. Let Q be a plane quadrangulation with n vertices so that each inner vertex is adjacent to at most 4 edges. The mixing time of the lattice walk on 2-orientations of Q satisfies $\tau(1/4) \in O(n^8)$.

- Define a tower Markov chain.

  Each step of the tower chain M_T can be simulated as a sequence of steps of the lattice walk M_2.

- Use a coupling argument to show that M_T is rapidly mixing.
- Use a comparison argument to show that M_2 is rapidly mixing.

The End



Thank you.