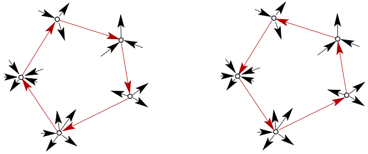
α-Orientations

Definition. Given $G = (V, E)$ and $\alpha : V \to \mathbb{N}$.
An α-orientation of G is an orientation with $\text{outdeg}(v) = \alpha(v)$ for all v.

- Reverting directed cycles preserves α-orientations.



Theorem. The set of α-orientations of a planar graph G has the structure of a distributive lattice.

Proof I: Essential Cycles

For the proof we assume that G is 2-connected.

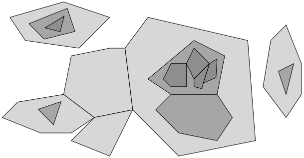
Definition.
A cycle C of G is an essential cycle if

- C is chord-free and simple,
- the interior cut of C is rigid,
- there is an α-orientation X such that C is directed in X.

Lemma.
C is non-essential $\iff$ C has a directed chordal path in every α-orientation.

Proof II

Lemma.
Essential cycles are interiorly disjoint or contained.



Lemma.
If C is a directed cycle of X, then X^C can be obtained by a sequence of reversals of essential cycles.

Lemma.
If $(C_1, .., C_k)$ is a flip sequence (ccw $\rightarrow$ cw) on X then for every edge e the essential cycles $C^{l(e)}$ and $C^{r(e)}$ alternate in the sequence.

Proof III: Flip Sequences

Lemma.
The length of any flip sequence (ccw $\rightarrow$ cw) is bounded and there is a unique α-orientation $X_{\min}$ with the property that all cycles in $X_{\min}$ are cw-cycles.

- $Y \prec X$ if a flip sequence $X \rightarrow Y$ exists.

Lemma.
Let $Y \prec X$ and C be an essential cycle. Every sequence $S = (C_1, \ldots, C_k)$ of flips that transforms X into Y contains the same number of flips at C.

Proof IV: Potentials

Definition. An α-potential for G is a mapping $\wp : \mathsf{Ess}_\alpha \to \mathbb{N}$ such that

- $|\wp(C) - \wp(C')| \leq 1$, if C and C' share an edge e.
- $\wp(C^{l(e)}) \leq \wp(C^{r(e)})$ for all e (orientation from $X_{\min}$)

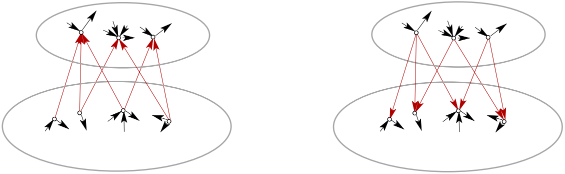
Lemma. There is a bijection between α-potentials and α-orientations.

Theorem. α-potentials are a distributive lattice with

- $(\wp_1 \vee \wp_2)(C) = \max\{\wp_1(C), \wp_2(C)\}$ and
- $(\wp_1 \wedge \wp_2)(C) = \min\{\wp_1(C), \wp_2(C)\}$ for all essential C.

A Dual Construction: c-Orientations

- Reorientations of directed cuts preserve flow-difference (#forward arcs − #backward arcs) along cycles.



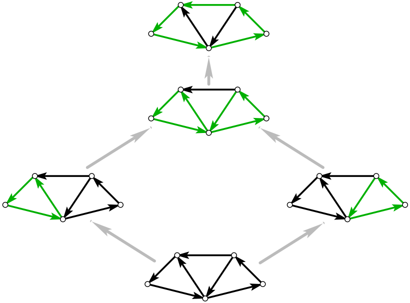
Theorem [Propp 1993]. The set of all orientations of a graph with prescribed flow-difference for all cycles has the structure of a distributive lattice.

- Diagram edge $\sim$ push a vertex ($\neq v_{\dagger}$).

Circulations in Planar Graphs

Theorem [Khuller, Naor and Klein 1993 **].**
The set of all integral flows respecting capacity constraints ($\ell(e) \leq f(e) \leq u(e)$) of a planar graph has the structure of a distributive lattice.

$0 \leq f(e) \leq 1$



- Diagram edge $\sim$ add or subtract a unit of flow in ccw oriented facial cycle.

Δ-Bonds

$G = (V, E)$ a connected graph with a prescribed orientation.
With $x \in \mathbb{Z}^E$ and C cycle we define the circular flow difference

$$\Delta_x(C) := \sum_{e \in C^+} x(e) - \sum_{e \in C^-} x(e).$$

With $\Delta \in \mathbb{Z}^{\mathcal{C}}$ and $\ell, u \in \mathbb{Z}^E$ define

$$\mathcal{B}_G(\Delta, \ell, u) = \left\{ x \in \mathbb{Z}^E \;:\; \Delta_x = \Delta \text{ and } \ell \leq x \leq u \right\}.$$

Δ-Bonds as Generalization

$\mathcal{B}_G(\Delta, \ell, u)$ is the set of $x \in \mathbb{Z}^E$ such that

- $\Delta_x = \Delta$ (circular flow difference)
- $\ell \leq x \leq u$ (capacity constraints).

Special cases:

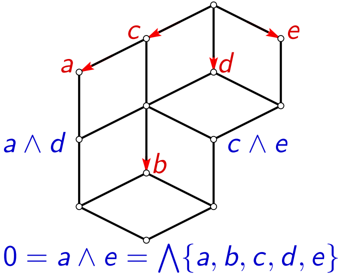
- c-orientations are $\mathcal{B}_G(\Delta, 0, 1)$
  ($\Delta(C) = \frac{1}{2}\big(|C^+| - |C^-| - c(C)\big)$).
- Circular flows on planar G are $\mathcal{B}_{G^*}(0, \ell, u)$
  (G^* the dual of G).
- α-orientations.

ULD Lattices

Definition. [Dilworth]
A lattice is an upper locally distributive lattice (ULD) if each element has a unique minimal representation as meet of meet-irreducibles.

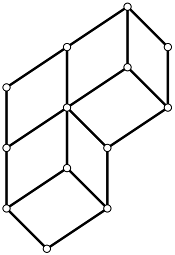
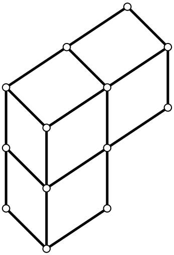
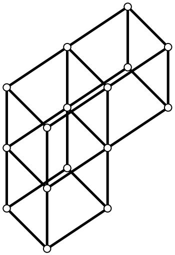
i.e., there is a unique mapping $x \to M_x$ such that

- $x = \bigwedge M_x$ (representation.)
- $x \neq \bigwedge A$ for all $A \subsetneq M_x$ (minimal).



ULD vs. Distributive

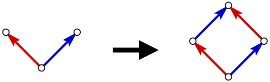
Proposition.
A lattice it is ULD and LLD $\iff$ it is distributive.

Diagrams of ULD lattices: A Characterization

A coloring of the edges of a digraph is a U-coloring iff

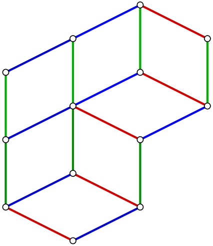
- arcs leaving a vertex have different colors.
- completion property:



Theorem.
A digraph D is acyclic, has a unique source and admits a U-coloring $\iff$ D is the diagram of an ULD lattice.

$\hookrightarrow$ Unique $\mathbf{1}$.

Examples of U-colorings



- Δ-bond lattices, colors are the names of pushed vertices. (Connected, unique $\mathbf{0}$).
- Chip firing game with a fixed starting position (the source), colors are the names of fired vertices.

More Examples

Some LLD lattices with respect to inclusion order:

- Subtrees of a tree (Boulaye '67).
- Convex subsets of posets (Birkhoff and Bennett '85).
- Convex subgraphs of acyclic digraphs (Pfaltz '71).
  (C is convex if with x, y all directed (x, y)-paths are in C).
- Convex sets of an abstract convex geometry (Edelman '80).
  (This is an universal family of examples).

Outline

A General problem: Sampling

- Ω a (large) finite set
- $\mu : \Omega \to [0, 1]$ a probability distribution, e.g. uniform distr.

Problem. Sample from Ω according to μ.

i.e., $\Pr(\text{output is } \omega) = \mu(\omega)$.

There are many hard instances of the sampling problem.
Relaxation: *Approximate sampling*

i.e., $\Pr(\text{output is } \omega) = \widetilde{\mu}(\omega)$ for some $\widetilde{\mu} \approx \mu$.

Applications of (approximate) sampling:

- Get hand on typical examples from Ω.
- Approximate counting.

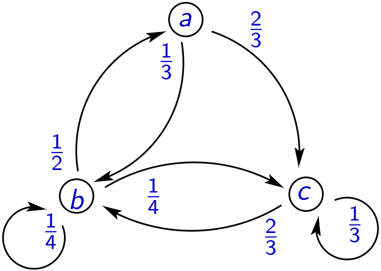
Preliminaries on Markov Chains

M transition matrix

- format $\Omega \times \Omega$
- entries $\in [0, 1]$
- row sums $= 1$ (stochastic)

Intuition:

$$\mathbf{M} = \begin{bmatrix} 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{2}{3} & \frac{1}{3} \end{bmatrix}$$



M specifies a random walk

Ergodic Markov Chains

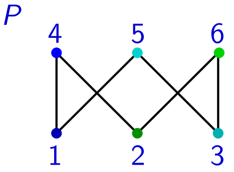
$\mathbf{M}$ is ergodic (i.e., irreducible and aperiodic)
$\implies$ multiplicity of eigenvalue 1 is one
$\implies$ unique π with $\pi = \pi\mathbf{M}$.

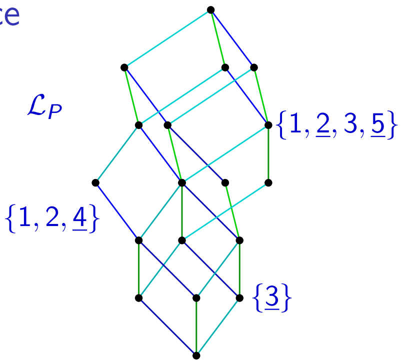
Fundamental Theorem.

$$\mathbf{M} \text{ ergoic} \implies \lim_{t\to\infty} \mu_0 \mathbf{M}^t = \pi.$$

$\mathbf{M}$ symmetric and ergodic
$\implies$ $\mathbf{M}^T \mathbb{1}^T = \mathbf{M}\mathbb{1}^T = \mathbb{1}^T$, hence $\mathbb{1}\mathbf{M} = \mathbb{1}$
$\implies$ π is the uniform distribution.

Example: Distributive Lattice





Lattice Walk
(A *natural* Markov chain on $\mathcal{L}_P$)

Identify state with downset D

- choose $x \in P$ & choose $s \in \{\uparrow, \downarrow\}$
- depending on s move to $D + x$ or $D - x$ (if possible)

Fact. The chain is ergodic and symmetric, i.e, π is uniform.

Mixing Time

$\mu_x^t = \delta_x \mathbf{M}^t$ the distrib. after t steps when start is in x

$\Delta(t) := \max(\|\mu_x^t - \pi\|_{VD} \ : \ x \in \Omega)$

$\tau(\varepsilon) = \min(t \ : \ \Delta(t) \leq \varepsilon)$

- $\tau(\varepsilon)$ is the mixing time.
- $\mathbf{M}$ is rapidly mixing $\iff$ $\tau(\varepsilon)$ is a polynomial function of $\log(\varepsilon^{-1})$ and the *problem size*.

Big Challenge. Find interesting rapidly mixing Markov chains

Example.

- Matchings (Jerrum & Sincair '88)
- Linear Extensions (Karzanov & Khachiyan '91 / Bubley & Dyer '99)
- Planar Lattice Structures, e.g. Dimer Tilings (Luby et al. '93)