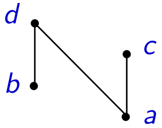
Linear Extensions

A linear extension of $P = (X, <)$ is a linear order L, such that

- $x <_P y \implies x <_L y$



d	c	d	d	c
c	d	b	c	d
b	b	c	a	a
a	a	a	b	b

Dimension of Orders I

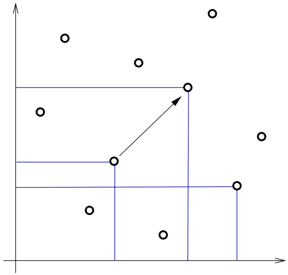
A family $\mathcal{L}$ of linear extensions is a realizer for $P = (X, <)$ provided that

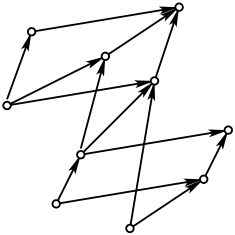
* for every incomparable pair (x, y) there is an $L \in \mathcal{L}$ such that $x < y$ in L.

The dimension, $\dim(P)$, of P is the minimum t, such that there is a realizer $\mathcal{L} = \{L_1, L_2 \ldots, L_t\}$ for P of size t.

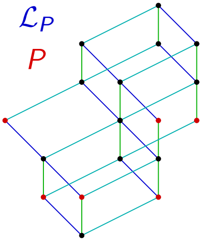
Dimension of Orders II

The dimension of an order $P = (X, <)$ is the least t, such that P is isomorphic to a suborder of $\mathbb{R}^t$ with the product ordering.





Dilworth's Imbedding Theorem (1950)



Theorem. $\dim(\mathcal{L}_P) = \text{width}(P)$.

- Let $C_1, \ldots, C_w$ be a chain partition of P.

  Imbed $\mathcal{L}_P$ in $\mathbb{R}^w$ by $I \to (|I \cap C_1|, \ldots, |I \cap C_w|)$.

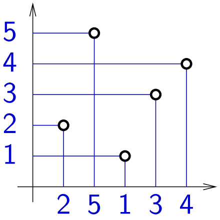
- If P contains an antichain A of size w,

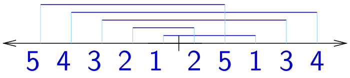
  then there is a Boolean lattice $\mathcal{B}_w$ in $\mathcal{L}_P$.

  Hence $\dim(\mathcal{L}_P) \geq \dim(\mathcal{B}_w) = w$.

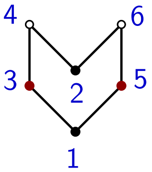
Small Dimension

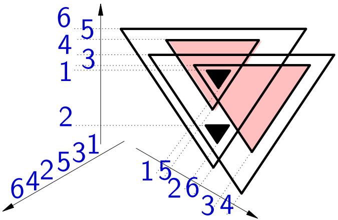
- Dimension 2: Containment orders of intervals.





- Dimension 3: Containment orders of triangles.

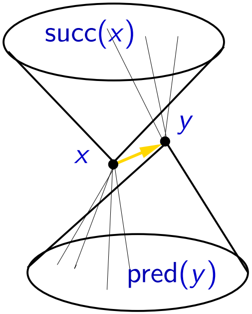




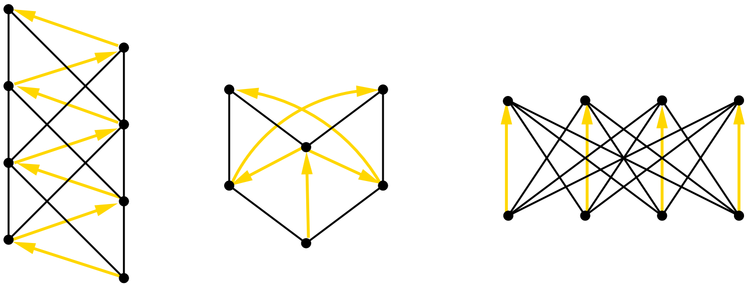
Critical Pairs

Definition. An incomparable pair (x, y) is critical if

- $a < x$ implies $a < y$.
- $y < b$ implies $x < b$.



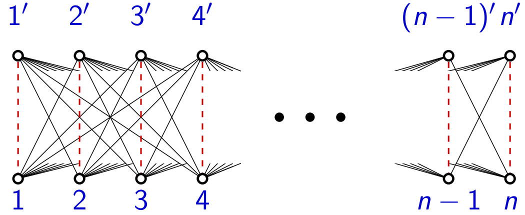
Critical Pairs



Proposition. A family $\mathcal{R}$ of linear extensions of P is a realizer of P $\iff$ $\mathcal{R}$ reverses all critical pairs.

Standard Examples

- Standard example of an n dimensional order:

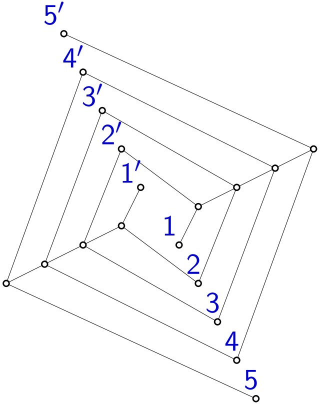


Dimension and Planarity

Theorem [Baker 1971]. If an order P has $\mathbf{0}$ and $\mathbf{1}$ and a planar diagram, then $\dim(P) \leq 2$.

Theorem [and Trotter and Moore 1977]. If an order P has $\mathbf{0}$ and a planar diagram, then $\dim(P) \leq 3$.

The dimension of an order P with a planar diagram can be unbounded (Kelly 1981).



Dimension beyond Planarity

Theorem [F., Li, and Trotter 2010 **].** The dimension of an order P of height ≤ 2 with a planar diagram is at most 4.

Theorem [Streib and Trotter 2014 **].** There is a function f such that $\dim(P) \leq f(h)$ for orders of height $\leq h$ with a planar cover graph.

Theorem [Joret, Micek, and Wiechert 2018 **].** There is a function $f_{\mathcal{C}}$ such that $\dim(P) \leq f_{\mathcal{C}}(h)$ for orders of height $\leq h$ whose cover graphs belong to a class $\mathcal{C}$ of graphs with bounded expansion. (This includes classes with a forbidden minor.)

Complexity

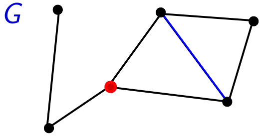
Theorem [Yannakakis 1982]. To test if a partial order has dimension $\leq k$ is NP-complete for all $k \geq 3$.
To test if a partial order of height 2 has dimension $\leq k$ is NP-complete for all $k \geq 4$.

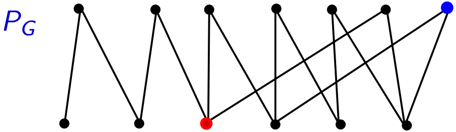
Theorem [F., Mustață, and Pergel 2014]. To test if a partial order of height 2 has dimension 3 is NP-complete.

Theorem [Chalermsook et al. 2013]. Unless NP $=$ ZPP there is no polynomial algorithm to approximate the dimension of a partial order with a factor of $O(n^{1-\varepsilon})$ for any $\varepsilon > 0$

Incidence Orders and Dimension

The incidence order P_G of G





Theorem [Spencer '72 / Trotter '80 / Hoșten und Morris '98].

$$\dim(K_n) = \dim(\mathbf{B}_n[1,2]) = \log\log n + (\tfrac{1}{2} + o(1)) \log\log\log(n)$$

$\dim(K_n) \leq$	2	3	4	5	6	7	8
$n \leq$	2	4	12	81	2646	1422564	229809982112

A Planarity Criterion

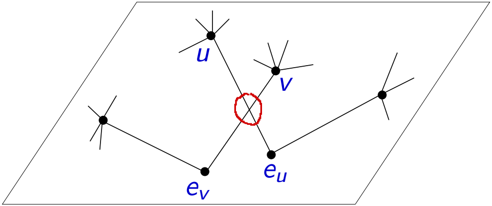
Theorem [Schnyder 1989].
A Graph G is planar $\iff$ $\dim(P_G) \leq 3$.

- $\dim(G) \leq 3 \implies G$ planar.

P_G in $\mathbb{R}^3$

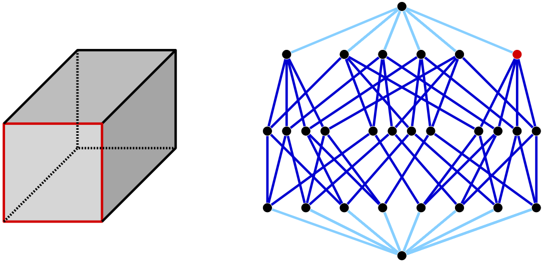
projection

$x + y + z = 1$



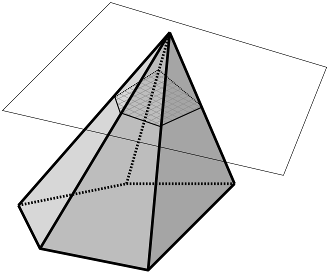
Dimension of Polytopes

Let $\mathcal{F}_P$ be the face lattice of polytope P.

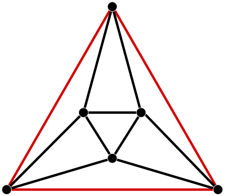


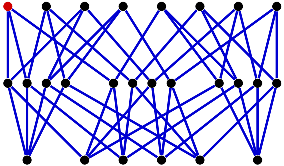
Dimension of Polytopes: Lower Bound

Theorem [Reuter 1990].
If P is a d-polytope, then $\dim(\mathcal{F}_P) \geq d+1$.



Dimension of 3-Polytopes





Theorem [Schnyder 1989].
If G is a plane triangulation with a face F, then

- $\dim(P_{VEF}(G \setminus F)) = 3$
- $\dim(P_{VEF}(G)) = 4$

Theorem [Brightwell+Trotter 1993].
If G is a 3-connected plane graph with a face F, then

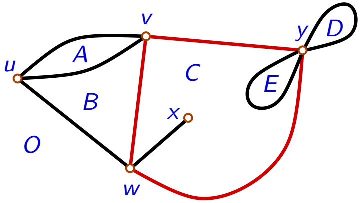
- $\dim(P_{VEF}(G \setminus F)) = 3$
- $\dim(P_{VEF}(G)) = 4$

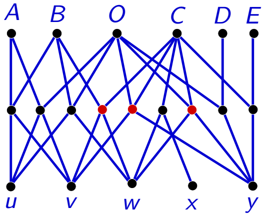
Dimension and Planar Graphs

Theorem [Schnyder 1989].
A Graph G is planar $\iff$ $\dim(P_G) \leq 3$.

Theorem [Brightwell+Trotter 1997].
If G is a plane multi-graph with loops, then

$$\dim(P_{VEF}(G)) \leq 4.$$



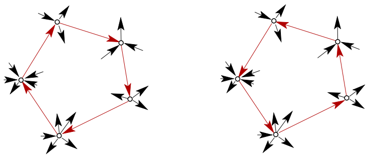


Outline

α-Orientations

Definition. Given $G = (V, E)$ and $\alpha : V \to \mathbb{N}$.
An α-orientation of G is an orientation with $\text{outdeg}(v) = \alpha(v)$ for all v.

- Reverting directed cycles preserves α-orientations.



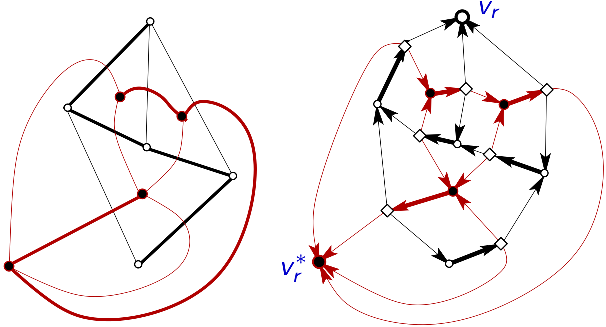
Theorem. The set of α-orientations of a planar graph G has the structure of a distributive lattice.

- Diagram edge $\sim$ revert a directed essential/facial cycle.

Example 1: Spanning Trees

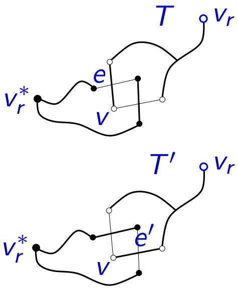
Spanning trees are in bijection with α_T orientations of a rooted primal-dual completion $\widetilde{G}$ of G

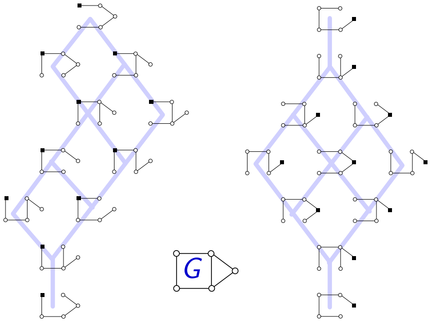
- $\alpha_T(v) = 1$ for a non-root vertex v and $\alpha_T(v_e) = 3$ for an edge-vertex v_e and $\alpha_T(v_r) = 0$ and $\alpha_T(v_r^*) = 0$.



Lattice of Spanning Trees

Gilmer and Litheland 1986, Propp 1993



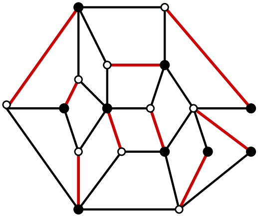
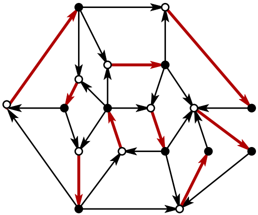


Example2: Matchings and f-Factors

Let G be planar and bipartite with parts (U, W). There is bijection between f-factors of G and α_f orientations:

- Define α_f such that $\text{indeg}(u) = f(u)$ for all $u \in U$ and $\text{outdeg}(w) = f(w)$ for all $w \in W$.

Example. A matching and the corresponding orientation.

Example 3: Eulerian Orientations

- Orientations with $\text{outdeg}(v) = \text{indeg}(v)$ for all v, i.e. $\alpha(v) = \frac{d(v)}{2}$