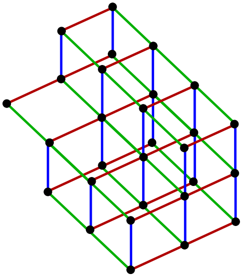
Order and lattices from graphs

Spring School SGT 2018
Seté, June 11-15, 2018



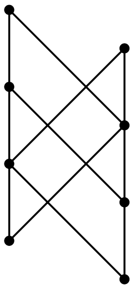
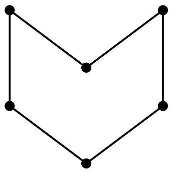
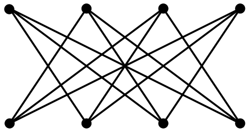
Stefan Felsner
Technische Universität Berlin

Outline

Finite Orders

$P = (X, <)$ is an order iff

- X finite set
- $<$ transitive and irreflexive relation on X.

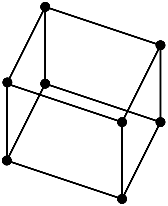
Lattices

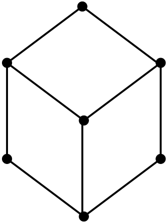
$P = (X, <)$ an order.

- Let $x \vee y$ be the least upper bound of x and y if it exists.
- Let $x \wedge y$ be the greatest lower bound of x and y if it exists.

$L = (X, <)$ is a finite lattice iff

- L is a finite order
- $x \vee y$ and $x \wedge y$ exist for all x and y.





Lattices - the algebraic view

$L = (X, \vee, \wedge)$ is a finite lattice iff

- X is finite and for all $a, b, c \in X$ and $\diamond \in \{\vee, \wedge\}$
- $a \diamond (b \diamond C) = (a \diamond b) \diamond c$ (associativity)
- $a \diamond b = b \diamond a$ (commutativity)
- $a \diamond a = a$ (idempotency)
- $a \vee (a \wedge b) = a$ and $a \wedge (a \vee b) = a$ (absorption)

Proposition. The two definitions of finite lattices are equivalent via:

$$(x \leq y \quad \text{iff} \quad x = x \wedge y) \quad \text{and} \quad (x \leq y \quad \text{iff} \quad x = x \vee y).$$

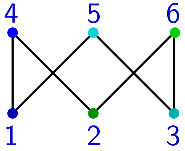
Distributive Lattice

A lattice $L = (X, \vee, \wedge)$ is a distributive lattice iff

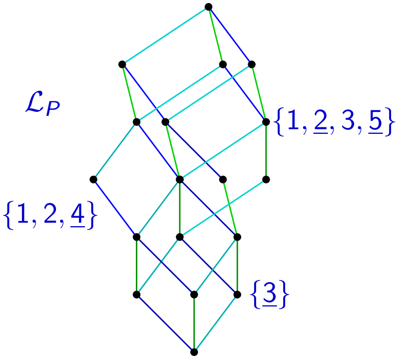
$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c) \quad \text{and} \quad a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$$

FTFDL. L is a finite distributive lattice $\iff$
there is a poset P such that that L is isomorphic to the inclusion order on downsets of P.

P



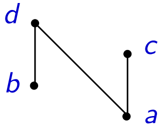
$\mathcal{L}_P$



Linear Extensions

A linear extension of $P = (X, <)$ is a linear order L, such that

- $x <_P y \implies x <_L y$



d	c	d	d	c
c	d	b	c	d
b	b	c	a	a
a	a	a	b	b

Dimension of Orders I

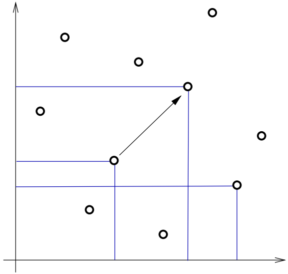
A family $\mathcal{L}$ of linear extensions is a realizer for $P = (X, <)$ provided that

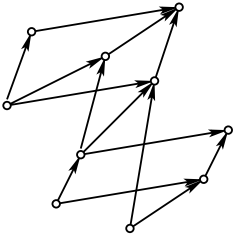
* for every incomparable pair (x, y) there is an $L \in \mathcal{L}$ such that $x < y$ in L.

The dimension, $\dim(P)$, of P is the minimum t, such that there is a realizer $\mathcal{L} = \{L_1, L_2 \ldots, L_t\}$ for P of size t.

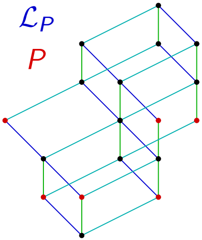
Dimension of Orders II

The dimension of an order $P = (X, <)$ is the least t, such that P is isomorphic to a suborder of $\mathbb{R}^t$ with the product ordering.





Dilworth's Imbedding Theorem (1950)



Theorem. $\dim(\mathcal{L}_P) = \text{width}(P)$.

- Let $C_1, \ldots, C_w$ be a chain partition of P.

  Imbed $\mathcal{L}_P$ in $\mathbb{R}^w$ by $I \to (|I \cap C_1|, \ldots, |I \cap C_w|)$.

- If P contains an antichain A of size w,

  then there is a Boolean lattice $\mathcal{B}_w$ in $\mathcal{L}_P$.

  Hence $\dim(\mathcal{L}_P) \geq \dim(\mathcal{B}_w) = w$.