1 Spring equilibrium – general setup

1.1 Informal introduction – setup

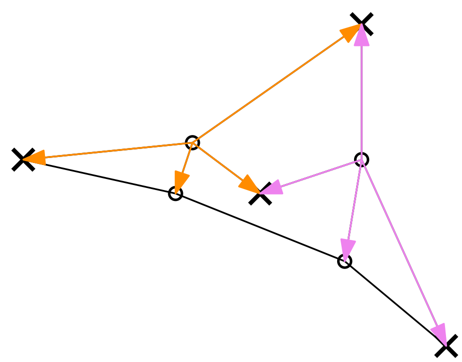

Figure 1: We'll always have a graph $G = (V, E)$ with weights on edges $\omega : E \to \mathbb{R}$. We'll be looking at straight line drawings of G on the plane, produced by some positioning of its vertices $p : V \to \mathbb{R}^2$. (And it will never be important that we work with $\mathbb{R}^2$ - absolutely everything we discuss works in $\mathbb{R}^d$ without any changes in reasoning, and all the proofs in fact is sufficient to conduct for 1d case, or "per coordinate"). We'll have some of the vertices of the graph designated as "fixed vertices", denoted $V^f \subset V$ – for the fixed vertices we'll prescribe the positioning p externally once and for all. The other vertices we'll be positioning in equilibrium – so that the vectors towards its neighbors, weighted with weights ω sum up to zero. On the figure we have vertices denoted with crosses as fixed. The vertices denoted with circles are "free" and shall be in equilibrium – for instance, the sum of orange vectors, scaled with weights assigned to the edges, shall be zero.

1.2 Definitions and examples

Let $G = (V, E)$ be a graph, let $\omega : E \to \mathbb{R}$ be an assignment of real weights to its edges, and let $p : V \to \mathbb{R}^2$ be a map, positioning vertices of G in the plane.

Definition 1 (it was not phrased in the lecture separately from definition2)**.** *We say that the vertex v_i of G is* in equilibrium *w.r.t. the weights ω if*

$$\sum_{v_j \in N(v_i)} \omega_{ij} \cdot (p(v_j) - p(v_i)) = 0, \tag{1}$$

where the summation goes over all the neighbours v_j of v_i.

Definition 2. *Assume V^f be some subset of vertices of G, that we'll call "fixed" vertices. The remaining vertices we refer to as "free vertices". We say that the system (p, ω, V^f) is in equilibrium if all the free vertices are in equilibrium.*

Remark 1. *Alternative way to rewrite the equilibrium condition: Eq. 1 is equivalent to*

$$\left(\sum_{v_j \in N(v_i)} \omega_{ij} \right) p(v_i) = \sum_{v_j \in N(v_i)} \omega_{ij} p(v_j) \tag{2}$$

and (as long as the denominator is non-zero) to

$$p(v_i) = \frac{1}{\sum_{v_j \in N(v_i)} \omega_{ij}} \sum_{v_j \in N(v_i)} \omega_{ij} p(v_j) \tag{3}$$

Equation 3 is probably the most canonical way imaginable to write down the idea "p_j is the weighted average of its neighbors".

Remark 2 (was not phrased at the lecture)**.** *From Eq. 3 it's also immediate, that when all weights are 1, every free vertex is in the barycenter of it's neighbors, thus it's also called "barycentric embedding".*

Example 1.

- *we fix exactly one point, $V^f = \{v_n\}$. It's trivial to see, that the only equilibrium position of such system is $p_i = p_n \quad \forall i$.*
- *we fix two points, $V^f = \{v_{n-1}, v_n\}$. It's simple to see that in the equilibrium position of such system all free points will lie on the segment $[p_{n-1}, p_n]$. [wasn't proven on the lecture. it's a trivial consequence of substatement of the Tutte theorem, that all free vertices are inside the convex hull of the fixed vertices – that was proven on Lecture 2.]*

1.3 Existence, uniqueness, and connection to Energy

Theorem 1. *Let $G = (V, E)$ be a connected graph, let $\omega : E \to \mathbb{R}_+$ be an assignment of positive real weights to its edges, let $V^f \subset V$ be a nonempty subset of "fixed" vertices, and let $p^f : V^f \to \mathbb{R}^2$ be a prescribed positioning of fixed vertices in the plane.*

Then there exists a unique extension of p^f to all the vertices of the graph, $p : V \to \mathbb{R}^2$ s.t $p|_{V^f} = p^f$, such that every "interior" vertex is in equilibrium w.r.t weights ω,

and this map p delivers the minimum of the Energy function

$$E(p) = \sum_{(v_i,v_j)\in E(G)} \omega_{ij}|p(v_i) - p(v_j)|^2,$$

over all maps p with the "boundary condition" $p|_{V^f} = p^f$

Remark 3.
- *The map p positions vertices of the graph to the plane, i.e. it's a map from a finite set of size $|V|$ to $\mathbb{R}^2$, and thus it can be though of as a point in $\mathbb{R}^{2|V|}$,*
- *Thus, function $E(p)$ can be though of as a function from $\mathbb{R}^{2|V|} \to \mathbb{R}$,*
- *To simplify the notation further, let us re-enumerate vertices of the graph so that the first k vertices are free, and the last $n - k$ vertices are fixed:*

$$V \setminus V^f = \{v_1, v_2, \ldots, v_k\}$$
$$V^f = \{v_{k+1}, \ldots v_n\}$$

- *And for the positioning of the vertices p we can use the notation*

$$p(v_i) =: p_i,$$

thus clarifying the very first point of this remark

$$p \equiv (p(v_1), p(v_2), \ldots, p(v_n)) = (p_1, \ldots p_n) \in \mathbb{R}^{2|V|}.$$

- *Finally, since p is preset by p^f on V^f, i.e. $p_j = p^f_j, j \in V^f$, it's natural to see function $E()$ as only the function of the free vertex positions:*

$$\bar{E}(p_1, \ldots p_k) := E(p_1, \ldots p_k, p^f_{k+1}, \ldots p^f_n)$$

Lemma 1. *The energy function $\bar{E}(p)$ is strictly convex on $\mathbb{R}^{2(|V|-|V^f|)}$.*

The proof was left as an exercise at the lecture.

Proof1: quick and dirty. The Hessian matrix of $\bar{E}(p)$ is irreducible, symmetric, diagonally dominated, with at least one row/column strictly diagonally dominated (it's actually the Laplacian with removed columns and rows corresponding to fixed vertices, see reduced Laplacian $\bar{L}$ in Sec. 1.4). Thus it is strictly positive definite, which proves strict convexity. We note that we used nonempty fixed set to say that at least one row/column is strictly diagonally dominated and connectedness of the graph to say that Hessian is irreducible, both were needed. □

Proof2: assume we know nothing. Since every summand $E_{ij}(p) = \omega_{ij}|p_i - p_j|^2$ is non-strictly convex (check in coordinates if in doubts), the whole energy function $E(p)$ is non-strictly convex. Thus we only need to show that the convexity is strict.

We prove by contradiction. Assume for some points $\bar{p}$ and $\bar{q}$ in $\mathbb{R}^{2(|V|-|V^f|)}$, the convexity is non-strict, i.e .

$$\bar{E}(\alpha\bar{p} + (1-\alpha)\bar{q}) = \alpha\bar{E}(\bar{p}) + (1-\alpha)\bar{E}(\bar{q}), \qquad \forall\alpha \in [0,1].$$

This by definition of $\bar{E}$ means that for points $p := (\bar{p}, p^f)$ and $q := (\bar{q}, p^f)$ in $\mathbb{R}^{2|V|}$

$$E(\alpha p + (1-\alpha)q) = \alpha E(p) + (1-\alpha)E(q), \qquad \forall\alpha \in [0,1]. \tag{4}$$

Rewriting above by the definition of E, on the l.h.s we have a coefficient in front of α^2 of

$$|p_i - p_j|^2 + |q_i - q_j|^2 - 2(p_i - p_j, q_i - q_j) = |(p_i - p_j) - (q_i - q_j)|^2 = |(p_i - q_i) - (p_j - q_j)|^2.$$

This coefficient must be zero since on the r.h.s of Eq. 4.we had no quadratic term, meaning that $p_i - q_i = p_j - q_j \forall i, j \in E$. Since the graph is connected it means that $p_i - q_i = C \forall i$ for some fixed vector $C \in \mathbb{R}^2$, i.e. p and q are parallel translation of each other. This however is impossible for non-zero C since the last $n - k > 0$ components of both p and q are the fixed points p^f, i.e. $p_j = q_j = p_j^f \forall j > k$. □

Lemma 2. *There exists $R > 0$ such that*

$$\bar{E}(0) < \bar{E}(\bar{p}) \qquad \forall\bar{p} : ||\bar{p}|| > R$$

Proof. This is trivial. □

Lemma 3. *The gradient of the function $\bar{E}$ is*

$$\nabla\bar{E} = 2\left(\tfrac{\partial\bar{E}}{\partial p_1}, \ldots, \tfrac{\partial\bar{E}}{\partial p_k}\right) \in \mathbb{R}^{2k},$$

with each of the partial derivatives being 2-dimensional vectors and being equal to exactly the l.h.s of the equilibrium condition at the corresponding point:

$$\frac{\partial\bar{E}}{\partial p_k} = 2\sum_{j\in N(k)} \omega_{jk}(p_k - p_j).$$

Proof.
- Approach1: just write everything in 1d coordinates, through x_i, y_i, and compute derivatives of quadratic functions,

- Approach2: it's a very useful meditative statement on its own, that if you have a function $f : \mathbb{R}^d \to \mathbb{R}$, which measures the squared distance to some fixed point, i.e.

$$f(p) = ||p - p_0||^2, \qquad \text{for some fixed } p_0,$$

then

$$\nabla f(p) = 2(p - p_0) \in \mathbb{R}^d$$

in a proper, coordinate-free understanding of the gradient, i.e.

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} f(p + \varepsilon h) - f(p)) = 2 < p - p_0, h >,$$

where $<, >$ is the d-dimensional dot product.

$\square$

Proof of Theorem 1. Since E is continuous, by Lemma 2 it has a local minimum strictly inside the ball of radius R. By Lemma 1, function $\bar{E}$ can have at most one local minima, and this local minima is then also the unique global minima. Finally, the equilibrium condition is equivalent to the gradient of $\bar{E}$ at point $\bar{p}$ being equal to zero, which is equivalent to point $\bar{p}$ being a local extremum. This finalises the proof. $\square$

1.4 Laplacian of the graph

Context: to a non-oriented weighted graph one can associate a matrix called Laplacian matrix.

Definition 3. *Let $G = (V, E)$ be a graph with n vertices ($|V| = n$), let $\omega : E \to \mathbb{R}$ assign weights to the edges of G. Then the Laplacian matrix is defined as an $n \times n$ matrix L_{ij} with*

$$L_{ij} = \begin{cases} \sum_{k \in N(i)} \omega_{ik} & \text{if } i = j \\ -\omega_{ij} & \text{if } (i,j) \in E(G) \\ 0 & \text{else.} \end{cases}$$

Observation 1.

- *Trivially, if the graph has no fixed vertices, then the condition for all points to be in equilibrium is equivalent to*

$$Lp = 0.$$

- *(Only briefly mentioned in the lecture) Matrix L is degenerate – the sum of all columns is trivially 0. Thus there exists a nontrivial solution to the equation $Lp = 0$, which corresponds to all vertices of the graph being positioned at the same point in the plane. One can show that for a connected graph this trivial solution is the only solution. This however is not the situation that interests us primarily.*

- *(Only briefly mentioned in the lecture) A way more interesting case is when V^f is nonempty. It's simple to check, that the condition for all "free" points to be in equilibrium then rewrites as*

$$\bar{L}\bar{p} = a,$$

where

  - *$\bar{L}$ is the Laplacian matrix, with removed columns and rows, corresponding to "fixed" vertices (V^f);*
  - *$\bar{p} = (p_1, \dots p_k)$ is the vector of positions of the "free" vertices ($V \setminus V^f$);*
  - *$a = (a_1, \dots a_k)$ with $a_i = \sum_{j \in N(i) \cap V^f} \omega_{ij} p_j$*

*Since the matrix L is symmetric and (strictly) diagonally dominated, the solution to Eq.**??** exists and is unique, giving us the desired embedding in equilibrium. This is an alternative proof of the (first part of) Theorem 1*

A A few useful remarks not explicitly said at the lecture

Remark 4. *1. If points are in equilibrium, then their projections to any subspace are in equilibrium.*

2. And the other way around, if projections into any d linear independent directions are in equilibrium, then the d dimensional points are.

3. In particular, points $p_i \in \mathbb{R}^2$ are in equilibrium iff their x coordinates are in equilibrium and their y coordinates are in equilibrium.

Remark 5. *In spirit of Remark 4, the function E can also be treated per coordinate. Indeed,*

$$||p_i - p_j||^2 = |x_i - x_j|^2 + |y_i - y_j|^2$$

and so

$$E(p_1, ..., p_k) = E_x(x_1, ..., x_k) + E_y(y_1, ..., y_k),$$

where

$$E_x(x_1, ...x_k) = \sum \omega_{ij}(x_j - x_i)^2$$
$$E_y(y_1, ...y_k) = \sum \omega_{ij}(y_j - y_i)^2$$

And one can easily see, that the minimisation of E can be as well performed independently per-coordinate.

B Very special setup – Tutte theorem

Definition 4. *Let $G = (V, E)$ be a planar 3-connected graph, let f^o be a face of G. Let $\omega : E \to \mathbb{R}_+$ be an assignment of strictly positive weights to edges of G, and let $p : V \to \mathbb{R}^2$ be such a positioning of vertices of G in the plane that*

- *The face f^o is drawn as a convex polygon P,*
- *Every vertex not belonging to the face f^o is in equilibrium w.r.t ω.*

Then the drawing p is called a "Tutte embedding" of G with weights ω and the outerface f^o embedded as P. We'll denote such embedding by $Tutte(G, \omega, f^o, P)$.

We'll call the vertices of G on the outerface f^o external*, and the vertices of G not belonging to the outerface –* internal*, Similarly, edges of f^o are called external, and the other edges – internal. Finally, all faces except for the outerface are referred to as internal faces.*

Theorem 2 (Tutte theorem)**.** *Every Tutte embedding is indeed a planar embedding of G, with every face embedded as a strictly convex polygon.*