Technische Universität Berlin
Institut für Mathematik

Winter 2025/26
Stefan Felsner, Robert Lauff

Graph Theory (DS II) - Sheet 13

Exercise 13.1.

Prove or disprove:

(a) Let G be a graph with $\chi(G) = k$. Then G has a k-coloring such that one of the color classes has size $\alpha(G)$.

(b) Let d_G be the average degree of G. Then $\chi(G) \leq d_G + 1$.

Exercise 13.2.

Given a triangle-free graph G_k with $\chi(G_k) \geq k$ and n vertices, let G_{k+1} have an independent vertex set X of $k(n-1)+1$ vertices and for all $Y \subset X, |Y| = n$ a copy of G_k, that is connected to Y by a perfect matching. Show that G_{k+1} is triangle-free and that $\chi(G_k + 1) \geq k + 1$.

Exercise 13.3.

Greedy coloring worst case:

(a) Show that there is a bipartite graph on $2n$ vertices and an order of the vertices, such that the Greedy algorithm uses n colors to color it instead of 2.

(b) Show that there is a planar graph on $2n$ vertices and an order of the vertices such that the Greedy algorithm uses more than n colors to color it.

Exercise 13.4.

Let there be some lines in the plane, such that no three intersect in a single point. Let the intersection points of these lines be the vertices of a graph G. Two such vertices are adjacent, if they are consecutive on one of the lines. Show that $\chi(G) \leq 3$.

Bonus Exercise

Consider the set of subsets of $\mathbb{N}$. This is a partially ordered set with the subset relation. What is its height? In other word, what is the cardinality of the largest set $\mathcal{S}$ of subsets of $\mathbb{N}$ such that for any two $A, B \in S$, either $A \subseteq B$ or $B \subseteq A$?