Technische Universität Berlin Winter 2025/26
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Graph Theory (DS II) - Sheet 12

Exercise 12.1.

Prove the following:

(a) For every 2-connected graph G with a pre-selected simple path P in it we can assign real numbers to the vertices of G such that they decrease along P, and every vertex except endpoints s, t of P has a neighbor with a number higher and a neighbor with a number lower than itself.

(b) For every 2-connected plane graph G with a pre-selected simple path P whose endpoints s, t are on the outer face we can assign real numbers to the vertices of G such taht they decrease along P and also along both paths connecting s to t along the outer face, and every vertex except s and t has a neighbor with a number higher and a neighbor with a number lower than itself.

Exercise 12.2.

Consider a closed polygonal chain drawn in the plane – with straight line segments of all nonzero lengths, but otherwise no restrictions on intersections or anything. We pick one direction in the plane as "vertical" and walk along our polygonal chain, measuring, how many times during one full cycle walk we change the direction of travel from "up" (w.r.t this chosen vertical direction) to "down" (or from "down" to "up"). Show that this number is exactly 2 for every choice of the "vertical direction" (except finitely many directions orthogonal to edges of the chain) if and only if the polyhedral chain is a boundary of a convex polygon.

Exercise 12.3.

Consider $n+1$ points in the plane, p, $p_1, p_2, \ldots, p_n$ and assume for simplicity that no 3 points are collinear. We pick one direction in the plane as "vertical" and start walking along the closed polygonal chain $p_1, ..., p_n, p_1$, measuring, how many times we switch from being "above" p to being "below" p (or vice versa) (above and below w.r.t the chosen "vertical" direction).

(a) Show that if this number if 2 for every choice of "vertical direction" (except finitely many), then the points p_{i-1} and p_{i+1} (indices taken modulo n) are on different sides of the line through p and p_i, for every $i = 1, \ldots, n$.

(b) Show that this number is 2 for every choice of "vertical direction" (except finitely many) if and only if the angles $p_i p_{i+1}$ all have the same orientation and add up to $\pm 2\pi$.

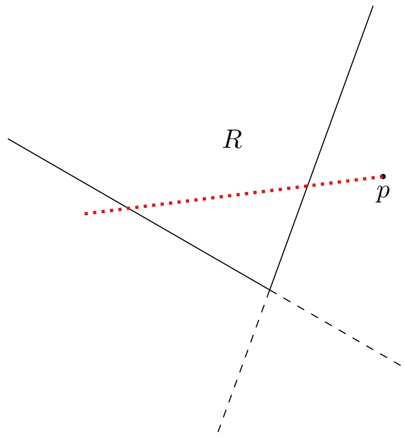

Figure 1: An illustration for the Bonus Exercise. The red dotted ray from p cuts of a region from R of some given area.

Exercise 12.4.

Let G be a connected graph (exceptionally, this exercise requires neither planarity nor 3-connectedness), and let $p\colon V \to \mathbb{R}^2$ be an (injective) embedding of vertices of G into the plane such that all vertices are in equilibrium w.r.t some assignment of non-zero weights $\omega\colon E \to \mathbb{R}$ to edges of G. Show that at least one edge is assigned a positive weight, and at least one edge is assigned a negative weight.

Bonus Exercise

Given two lines ℓ_1 and ℓ_2 in the plane. They cut the plane into 4 regions. Pick one of the regions R and place a point p in one of the regions adjacent to R (i.e. the region in which p is and R are separated by only one line). Given some area $A \in \mathbb{R}_{\geq 0}$, there exists a unique ray shooting from p which cuts of a triangle of area A from R. Show that this ray can be constructed using only a straight edge and compass. The area A is given as a line segment of length A somewhere on the plane, so that you can set your compass to that distance.