Technische Universität Berlin Winter 2025/26
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Graph Theory (DS II) - Sheet 7

Exercise 7.1.

(a) Find a graph G whose parameters $\alpha(G)$[a], $\omega(G)$[b], $\chi(G)$[c] and $\theta(G)$[d] are pairwise different.

(b) Find $k, n \in \mathbb{N}$ such that $T_k(n)$ is the smallest Turán graph (with respect to the number of edges) that contains the Petersen graph as a subgraph

[a] An independent set is a set of vertices such that no two are connected by an edge and $\alpha(G)$ is the size of a maximum independent set.

[b] A clique is a set of vertices such that any two are connected by an edge and $\omega(G)$ is the size of a maximum clique.

[c] $\chi(G)$ is the size of a minimum partition of the vertices of G into independent sets.

[d] $\theta(G)$ is the size of a minimum partition of the vertices of G into cliques.

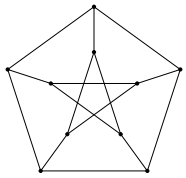

Figure 1: The Petersen graph

Exercise 7.2.

Let G be a graph. The Wei inequality says that $\alpha(G) \geq \sum_v \frac{1}{\deg(v)+1}$. Characterize the graphs for which equality holds.

Exercise 7.3.

Given a set of vectors $v_1, \ldots, v_n \in \mathbb{R}^d$ with $\|v_i\| \geq 1$. What is the maximum number of pairs v_i, v_j such that $\|v_i + v_j\| < 1$?

Hint: Triangles.

Exercise 7.4.

A directed graph is a graph where each edge (x, y) is directed $x \to y, y \to x$ or both, but without multi-edges or loops. A oriented graph is a directed graph where there can only be at most one directed edge between any two vertices.

(a) What is the maximum number of directed edges in a n-vertex directed graph without a cyclically directed triangle ($x \to y \to z \to x$)?

(b) What is the maximum number of directed edges in a n-vertex oriented graph without a cyclically directed triangle ($x \to y \to z \to x$)?

Bonus Exercise

Alice, Bob, Charlie and you are in a bar. You know that they are a mathematician, a physicist and an engineer, but you forgot who is what. Unfortunately for you, they agreed to mess with you a bit, so they decided to only answer your questions with foo or bar. The mathematician decided on some convention for which word means yes and which means no, so naturally, the physicist decided to use the opposite convention. The engineer had trouble following the plan, so they decided to just answer every question randomly with foo or bar. Because the music in the bar is so loud, you can only ever talk to one of the three at a time. How many questions do you need to ask in total to determine the professions of all three?