Technische Universität Berlin Winter 2025/26

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Graph Theory (DS II) - Sheet 6

Exercise 6.1.

Let $\text{ex}(n, H, F)$ be the maximum number of copies of H in an n-vertex F-free graph G. Note that $\text{ex}(n, K_2, F) = \text{ex}(n, F)$. Show that $\text{ex}(n, C_5, C_3) \geq \lfloor \frac{n}{5} \rfloor^5$.

Exercise 6.2.

Use the Matrix-Tree-Theorem to

(a) prove Cayley's formula.

(b) determine the number of spanning trees of $K_{m,n}$.

Exercise 6.3.

Let G be an undirected graph and $\vec{G}$ an orientation of G, that is, a graph that contains exactly one of (u, v) or (v, u) for each $\{u, v\} \in E(G)$. Let N be the vertex-edge incidence matrix of $\vec{G}$ and let $\hat{N}$ be N without its first row. For any $F \subset E$ we will write $\hat{N}(F)$ for the matrix with the columns of $\hat{N}$ that correspond to F. Let $F \subset E$ with $|F| = n - 1$ be given. When does

$$\det\left(\hat{N}(F)\right)^2 = 1$$

hold? What else could it be?

Exercise 6.4.

A universal de Bruijn sequence for n is an infinite sequence of symbols $a_1, a_2, \ldots$ from the infinite alphabet $\mathbb{N}_0$, such that for all m the first m^n symbols form a Memory Wheel, that is a de Bruijn sequence, for words of length n over the alphabet $\{0, \ldots, m - 1\}$. Show, that there are universal de Bruijn sequences for all $n \in \mathbb{N}$.

Exercise 6.5.

How many Hamilton cycles does the graph from Figure 1 have?

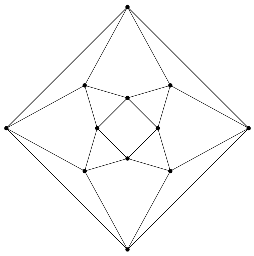

Figure 1: The graph

Bonus Exercise

You and a group of 9 other people have been captured by aliens would really like to eat you. However, they follow a law which forbids them from eating intelligent lifeforms. Therefore, they devise a game for you to play to determine whether eating you is ok or not. They tell you the following rules. In a few minutes, you will be lined up in a row and every person will get a hat on their head which is either black or white. A person cannot possible see their own hat and any communication will be impossible due to secret alien technology. Then, going from back to front, the aliens will ask each person after the other about the color of their hat. If you can come up with a strategy which guarantees that 9 out of 10 people guess the correct color, you will not be eaten. What is that strategy?