

Graph Theory (DS II) - Sheet 5

Exercise 5.1.

- (a) Let $G = (V, E)$ be a connected graph with exactly 2 vertices of odd degree. Show that G admits an Euler path.
- (b) Show that in every connected graph, there is a walk that contains every edge exactly twice.
- (c) Let G be a Eulerian digraph which is not a directed cycle. Show that G admits an even number of Euler cycles.

Exercise 5.2.

Prove that the graph in Figure 1 is not Hamiltonian.

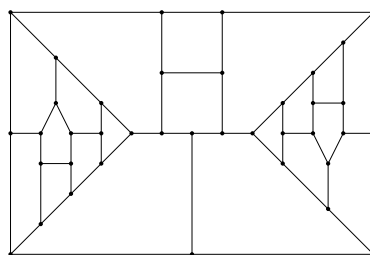


Figure 1: The smallest 3-regular 3-connected graph which is not Hamiltonian.

Exercise 5.3.

Gray codes:

- (a) Consider the standard reflected Gray code (SRGC)^a. Given a bitvector $b \in \{0, 1\}^n$, in which position does b occur in the SRGC of order n ? Given a position $p \in [2^n]$, which bitvector is in position p of the SRGC of order n ?
- (b) Consider the Steinhaus-Johnson-Trotter (SJT) Gray code. Given a permutation π of $[n]$, in which position does π occur in the SJT Gray code of order n ? Given a position $p \in [n!]$, which permutation is in position p of the SJT Gray code of order n ?

^aIn the lecture this might have been called the revolving doors algorithm.

Exercise 5.4.

Consider the set of set partitions of $[n]$ as vertices. Given a partition, order its sets by their minima in ascending order. A flip takes a value i from one of the sets of a partition and moves it to a neighbouring set in this order. Find a bijection between set partitions and certain permutations and use Algorithm J to prove that this flip graph has a Hamilton cycle.

Exercise 5.5.

Let T be a tree. Let G be the graph that contains a vertex for every vertex-edge incidence, that is for every pair $(v, e) \in V(T) \times E(T)$, such that $v \in e$, and an edge whenever two such incidences share the vertex or the edge, more formally

$$E(G) = \{((v, e), (v', e')) \in (V(T) \times E(T))^2 \mid v = v' \vee e = e'\}.$$

Prove that its square G^2 has a Hamilton cycle.

Bonus Exercise

Let there be $4n$ stones with the respective weights $1, 2, \dots, 4n$. Every stone has one of n colors, and there are exactly 4 stones of each color. Show that it is possible to split the stones into two parts such that

- the parts have the same weight, and
- both parts contain precisely 2 stones of each color.

Hint: The solution involves Euler cycles.