Technische Universität Berlin Winter 2025/26
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Graph Theory (DS II) - Sheet 3

Exercise 3.1.

Let $K_{n,m}$ be the complete bipartite graph with partition classes of size n and m respectively.

(a) How many spanning trees does $K_{2,n}$ have? How many up to isomorphism?

(b) How many spanning trees does $K_{3,n}$ have? How many up to isomorphism?

(c) How many spanning trees does $K_{m,n}$ have?[a]

[a]Hint: Consider the proof by Clarke from the lecture.

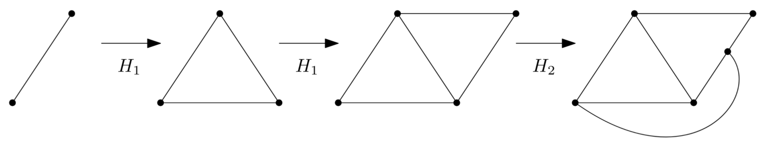

Figure 1: The two Henneberg steps.

Exercise 3.2.

A graph G is a *Laman*-graph if it has $2|V(G)|-3$ edges and each of its subgraphs H with at least 2 vertices has at most $2|V(H)|-3$ edges.

(a) Show that every Laman-graph can be obtained from K_2 by a sequence of so- called *Henneberg steps*: Either add a vertex of degree 2 adjacent to any two vertices of the graph (H_1) or replace an edge connecting two vertices u and v by a vertex x adjacent to u, v and any third vertex (H_2). See Figure 1.

(b) Show that in any Laman-graph, there are two spanning trees which share exactly one edge.

Exercise 3.3.

A connected graph with exactly one simple cycle is called a *pseudotree*. Prove that any two of these properties are equivalent:

(a) G is a pseudotree.

(b) G has n edges.

(c) There is an edge e of G such that $G - e$ is a tree.

Exercise 3.4.

Let G be a graph and let $F \subset E$ be a subset of its edges.

(a) F can be extended to an element of the cut space if and only if F contains no odd cycle.

(b) F can be extended to an element of the cycle space if and only if F contains no odd cut.

Exercise 3.5.

Prove the following equation by counting spanning trees of K_n which contain a fixed edge.

$$2n^{n-3} = \sum_{k\geq 0} \binom{n-1}{k} (n-k-1)^{n-k-3}(k+1)^{k-1}.$$

Bonus Exercise

Given n blue points on a circle. How many red points must be added such that there is no triangle with distinct blue points as vertices which does not contain a red point? **Hint:** Draw an edge between any pair of blue points and consider the regions.