Technische Universität Berlin Winter 2025/26
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Graph Theory (DS II) - Sheet 2

Dear Students, please have a look at the Seminar offered by our research group this semester. At the moment there might not be enough people for the seminar to take place, which would be a shame. We usually organize it like a small conference in a youth hostel near Berlin, which is a lot of fun. If you are interested in joining, write an email with a list of topics (see the seminar page) you like to Prof. Felsner.

Exercise 2.1.

(a) Show that $(d_1, \dots, d_n)$ is the degree sequence of a tree, if and only if $\sum_{i=1}^{n} d_i = 2n - 2$.

(b) Let $d_1 < \ldots < d_k$ be natural numbers. Show that there exists a graph G on $d_k + 1$ vertices such that $\{d_1, \dots, d_k\}$ is the set of degrees of G.

Exercise 2.2.

(a) What is the connectivity of the d-dimensional hypercube?

(b) Show that a k-connected graph on n vertices has at least $\lceil \frac{nk}{2} \rceil$ edges.

(c) Let $G = (V, E)$ be a k-connected graph and let $s_1, \dots, s_k, t_1, \dots, t_k \in V$ be pairwise distinct. Show that there are paths from s_i to t_i for $i = 1, \dots, k$ which are pairwise vertex-disjoint.

Exercise 2.3.

(a) Prove that $\operatorname{girth}(G) \leq 2 \operatorname{diam}(G) + 1$ for every graph which has at least one cycle.

(b) Show that the inequalities $\operatorname{rad}(G) \leq \operatorname{diam}(G) \leq 2 \operatorname{rad}(G)$ are both tight (not necessarily for the same graph G). Present graphs of arbitrarily high radius for both.

Exercise 2.4.

Let T be a tree on $n \geq 3$ vertices and let $x_i = |\{v \mid \deg(v) = i\}|$.

(a) Show that $\sum_{i=3}^{n-1} (i-2) x_i = x_1 - 2$.

(b) How many different trees (up to isomorphism) without vertices of degree 2 and with exactly 5 leaves are there?

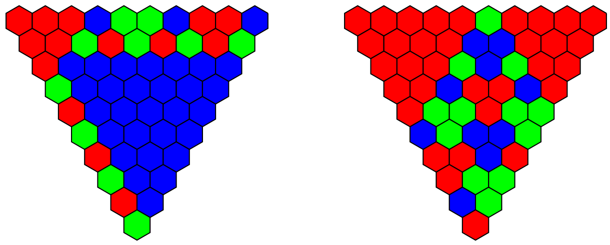

Figure 1: Two properly coloured hexagonal triangles.

Exercise 2.5.

Given two graphs $G = (V_G, E_G)$ and $H = (V_H, E_H)$, their cartesian product $G \times H$ is the graph with $V_{G\times H} = V_G \times V_H$ and

$$E_{G\times H} = \{((a,b),(c,d)) \mid a = c \text{ and } (b,d) \in E_H \text{ or } b = d \text{ and } (a,c) \in E_G\}.$$

Show that

(a) $\kappa(G \times H) \geq \kappa(G) + \kappa(H)$, where κ denotes the connectivity of a graph.

(b) $\operatorname{diam}(G \times H) = \operatorname{diam}(G) + \operatorname{diam}(H)$.

Bonus Exercise

Let x and y be two vertices of the d-dimensional hypercube H_d with distance d, i.e. they are antipodal. Let $B_r(x)$ and $B_r(y)$ be the sets of vertices of H_d with distance at most r from x and y, respectively. For $r < \frac{d}{2}$, determine the number of vertex disjoint paths from $B_r(x)$ to $B_r(y)$.

Hint: Symmetric chain decomposition.

Bonus Exercise

A *hexagonal triangle* is an arrangement of hexagons into a triangle as in Figure 1. The number of hexagons in the first row is the *order* of the triangle. Given a colouring of the hexagons in the first row with the colours red, green and blue, we extend this colouring to the rows below with the following rule. Given a hexagon which is not yet coloured, but its two adjacent hexagons in the row above are. Then, if the two hexagons above are of the same colour, we use that colour. Otherwise, we use the colour which is not present in the two hexagons above. A hexagonal triangle is called *properly coloured*, if it was coloured by this rule. For some n, we observe the following property about all properly coloured hexagonal triangles of order n: The colouring rule also applies to the three corners, i.e. if the two corners in the first row are of the same colour, then the corner that is the last row is also of that colour, and if they are not of the same colour, then the bottom corner is of the missing colour. For which n does this property always hold?