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Graph Theory (DS II) - Sheet 1

Exercise 1.1.

(a) Does a simple graph with at least two vertices exist, such that the vertices have pairwise distinct degrees?

(b) A k-*regular*-graph is a graph whose vertices all have degree k. Classify the 3-regular graphs with $1, 2, \ldots, 8$ vertices?

Exercise 1.2.

Let $G = (V, E)$ be a graph with $V = [n] := \{1, \ldots, n\}$. Prove that:

(a) If G contains a circle[a], then G contains a cycle[b].

(b) If G contains a circle of odd length, then G contains a cycle of odd length.

(c) If G contains no cycle of even length, then $|E| \leq \frac{3}{2}(n-1)$. Is this also true if G contains no odd cycle?[c]

[a] Here, a circle is a closed walk, i.e. it ends in the vertex it started, but can use vertices and edges multiple times.
[b] A cycle is only allowed to use each vertex and edge at most once.
[c] Recall that a cycle free graph with n vertices has at most $n-1$ edges.

Exercise 1.3.

(a) If u and v are the only vertices of odd degree in G, then there is a walk from u to v in G.

(b) A graph has a homomorphism to K_2 if and only if it is a subgraph of a complete bipartite graph.

(c) The following are equivalent:

- G has a homomorphism to K_2,
- The vertex set of G can be partitioned into two sets X and Y, such that the induced graphs $G[X]$ and $G[Y]$ are discrete, i.e. they do not contain any edges.
- G contains no odd cycle.

Any of these are a definition of bipartite graphs.

Exercise 1.4.

(a) Present a collection $A_1, \ldots, A_{n+1} \subset [n]$ of distinct sets, such that there exists no $x \in [n]$ with the property that $A_1 + x, \ldots, A_{n+1} + x$ are still distinct.

(b) Prove that for any collection $A_1, \ldots, A_{n-1} \subset [n]$ of distinct sets, there exist $x \neq y \in [n]$ such that $A_1 + x + y, \ldots, A_{n-1} + x + y$ are still distinct.

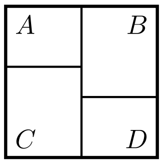

Figure 1: The rectangulation of the square from the Bonus Exercise.

Bonus Exercise

Consider the rectangulation of the square given in Figure 1, cut into the four regions A, B, C and D. Let P be a finite set of points in the square such that no two points have the same x- or y-coordinates. Given $a, b, c, d \in \mathbb{N}$ with $a + b + c + d = |P|$, show that one can slide the three segments of the rectangulation[a], such that a, b, c and d are exactly the number of points from P in A, B, C and D, respectively.

[a] The horizontal ones are slid up and down and the vertical one is slid left and right. When the vertical one is slid to the left and right, the horizontal ones are lengthened or shortened so that they sill touch the vertical one and the square is still cut into four regions.