8. Practice sheet for the lecture: **Felsner/ Wesolek**
Graph Theory (DS II) 05. December 2023
Due dates: 12./14. December
`https://page.math.tu-berlin.de/~felsner/Lehre/dsII23.html`

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(1) Prove Cayleys formula using the matrix-tree-theorem.

(2) In this exercise you will give an alternative proof of the matrix-tree-theorem. Fix an orientation $\vec{G}$ of a graph G. The oriented incidence matrix $W = W(\vec{G})$ is a $|V(\vec{G})| \times |E(\vec{G})|$ matrix such that for $e = (u, v)$ it holds that $W(u, e) = -1$, $W(v, e) = 1$ and $W(z, e) = 0$ if $z \neq u, v$.

 (a) Let $L = L(G)$ be the Laplacian of G and L' be the matrix obtained from L by deleting the first row and first column. Show that $L = WW^T$ and $L' = (W')(W')^T$ where W' is the matrix obtained from W by deleting the first row.

 (b) Let S be a subset of $n - 1$ edges of $\vec{G}$. Let W'_S be the matrix obtained from W' by deleting all columns except for those in S. Show that $\det(W'_S) = 1$ or $\det(W'_S) = -1$ if S corresponds to a spanning tree of G, and $\det(W'_S) = 0$ otherwise.

 (c) Use the Cauchy-Binet formula to show that $det(L')$ is equal to the number of spanning trees in G.
 The Cauchy-Binet formula states that for two $n \times m$ matrices A, B with $m \geq n$ it holds that

$$\det(AB^T) = \sum_{S \subset [m], |S| = n} \det(A_S) \det(B_S),$$

 where A_S is the matrix obtained from A by deleting all columns except for those in S (equivalently for B_S).

(3) A *universal de Bruijn sequence for n* is an infinite sequence of symbols $a_1, a_2, \ldots$ from the infinite alphabet $\mathbb{N}_0$ such that for all m the first m^n symbols form a Memory Wheel over the alphabet $\{0, \ldots, m - 1\}$, that is, a de Bruijn sequence for words of length n over the alphabet $\{0, \ldots, m - 1\}$. Show, that there are universal de Bruijn sequences for all $n \in \mathbb{N}, n \geq 2$. (Hint: Find a way to extend the sequence for the first m^n symbols to the first $(m + 1)^n$ by modelling the problem as an Euler cycle problem on a suitable directed graph.)

(4) How many possible ways are there to draw the "Haus vom Nikolaus" (See Figure 1)?

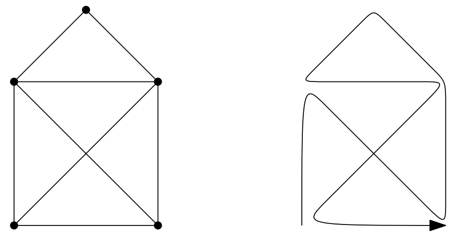

Figure 1: One of the ways of drawing the "Haus vom Nikolaus" without lifting the pen off of the paper while drawing nor drawing any edge twice, a popular German children's game