5. Practice sheet for the lecture: **Felsner/ Wesolek**
Graph Theory (DS II) 14. November 2023

Due dates: 21./23. November
`https://page.math.tu-berlin.de/~felsner/Lehre/dsII23.html`

(1) Prove that every connected graph has a walk that uses every edge exactly twice.

(2) Let $G = (V, E)$ be a graph und $F \subset E$ a subset of the edges. Show the following:

   (a) F can be extended[1] to an element of the cut space $S(G)$ if and only if F does not contain an odd cycle.

   (b) F can be extended to an element of the cycle space $Z(G)$ if and only if F does not contain an odd cut. [A cut is an element from the cut space, hence an induced cut.]

(3) Show that for any tree T with t edges, $\frac{(t-1)}{2}n - o(n) \leq ex(n, T)$. If $T = K_{1,t}$, show that the lower bound is correct (up to $o(n)$).

Next week: Show that $ex(n, T) \leq (t-1)n$.

(4) Let G be a graph on the vertex set $\{1, \ldots, n\}$ and let d_i be the degree of vertex i. Suppose

$$\sum_{i=1}^{n} \binom{d_i}{2} > (m-1)\binom{n}{2}.$$

   Show that G contains $K_{2,m}$ as a subgraph.

(5) Let $ex(n, H, F)$ be the maximum number of copies of H in an n-vertex F-free graph G. Note that $ex(n, K_2, F) = ex(n, F)$. Show that $ex(n, C_5, C_3) \geq \lfloor \frac{n}{5} \rfloor^5$.

[1] Which means edges can be added.