4. Practice sheet for the lecture: **Felsner/ Wesolek**
Graph Theory (DS II) 05. November 2023
Due dates: 14./16. November
`https://page.math.tu-berlin.de/~felsner/Lehre/dsII23.html`

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(1) A connected graph with at most one simple cycle is called a *pseudotree*. Prove that any two of these properties imply the third one:

- G is a pseudotree.
- G is connected and has n edges.
- There is an edge $e \in E(G)$ such that $G - e$ is connected.

(2) Given a graph $G = (V, E)$ and two spanning trees $T_1 = (V, E_1)$ and $T_2 = (V, E_2)$ of G, show that for every edge $e_1 \in E_1 \setminus E_2$ there is an edge $e_2 \in E_2 \setminus E_1$ such that $T_1' = (V, (E_1 \setminus \{e_1\}) \cup \{e_2\})$ and $T_2' = (V, (E(T_2) \setminus \{e_2\}) \cup \{e_1\})$ are also both spanning trees of G.

(3) Let X be a set of n points on a circle. Let D_n be a drawing of K_n on X with straight line edges. A non-crossing spanning path P_n of D_n is a spanning path in the corresponding K_n such that no edges of P_n cross in D_n. Show that if n is even, the edges of K_n can be partitioned into non-crossing spanning paths.

(4) Let X be a set of n points on a circle and T be a non-crossing straight line tree on the point set X. That is, the edges of T are straight lines and none of the edges cross. A *flip* in T is a removal of an edge $e \in T$ and an addition of an edge $e' \notin T$ such that $T' = T \setminus e \cup e'$ is a non-crossing straight line tree. Let T_1, T_2 be two non-crossing straight line trees on X. Show that there are at most $2n - 4$ flips needed to transform T_1 into T_2 for $n \geq 3$.

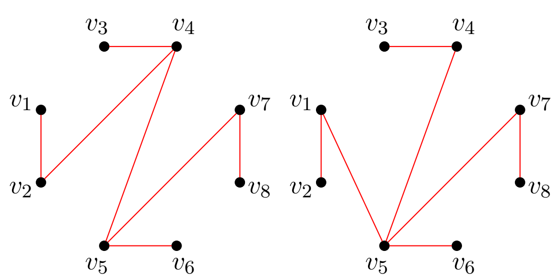

Figure 1: A flip.

(5) Let $K_{m,n}$ be the complete bipartite graph with parts $\{1', \dots, m'\}$ as well as $\{1, \dots, n\}$.

(a) How many spanning trees does $K_{2,n}$ have? How many non-isomorphic ones?

(b) How many spanning trees does $K_{3,n}$ have? How many non-isomorphic ones? [This is a rounded polynomial but a not fully simplified sum is acceptable.]

(c) Let $m \leq n$. How many spanning trees does $K_{m,n}$ have? [Hint: Clarke's proof of the Cayley formula can be adapted to $K_{m,n}$.]