12. Practice sheet for the lecture: **Felsner/ Wesolek**
Graph Theory (DS II) 16. January 2024
Due dates: 23./25. January
https://page.math.tu-berlin.de/~felsner/Lehre/dsII23.html

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(1) Adapt the random construction of Moon to obtain a drawing of $K_{n,m}$ with expected number of crossings equal to $\frac{1}{4}\binom{n}{2}\binom{m}{2}$.

(2) Find lower bounds on the crossing number of K_n.

  (a) Show that the crossing number of K_6 is 3.

  (b) Find a lower bound for K_n based on the fact that the crossing number of K_5 is 1.

  (c) Improve your bound from (b) by using[1] that the crossing number of K_{12} is 150.

(3) For any sets P of n points and L of m lines in the plane, let $I(P, L)$ denote the set of incidences, that is $I(P, L) = \{(p, \ell) \in P \times L \mid p \in \ell\}$. Prove that $|I(P, L)| \leq 4\max(n^{2/3}m^{2/3}, n) + m$. [Hint: Make a sketch and find a graph with n vertices and $I(P, L) - m$ edges.]

(4) Suppose you are given a triangulation and three vertices v_1, v_2, v_n that lie on the outer face. Construct a *canonical ordering* $v_1, \ldots, v_n$ of the vertices, that is, an ordering such that:

- the graph drawing D_k induced by $\{v_1, \ldots, v_k\}$ is internally triangulated for $k \in \{3, \ldots, n\}$,
- v_1v_2 is on the outer cycle of D_k for all $k \in \{3, \ldots, n\}$,
- v_{k+1} is on the outer face of G_{k+1} and its neighbours appear consecutively along the outer face of G_k for all $k \in \{3, \ldots, n-1\}$.

(5) For each integer $n \geq 3$ find an outerplanar graph which contains all outerplanar graphs of order n as subgraphs.

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[1] without proof