10. Practice sheet for the lecture: **Felsner/ Wesolek**
Graph Theory (DS II) 19. December 2023
Due dates: 09./11. January
https://page.math.tu-berlin.de/~felsner/Lehre/dsII23.html

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(1) A graph is called outerplanar if it has a plane drawing in which every vertex lies on the boundary of the outer face. Show that every outerplanar graph has degeneracy at most 2. [Hint: https://www.sfu.ca/~agwesole/Hinweis4]

(2) Let D be a drawing of an n-vertex graph G with c edge-crossings (where we assume no three edges cross in one point). Suppose G has minimum degree 7. Show that $c \geq \frac{n}{2} + 6$.

(3) A planar graph is a *triangulation*, if a drawing without crossings of the graph exists, such that every face has degree 3 (even the outer face). Show that:

(a) Every planar graph on $n \geq 3$ vertices is a subgraph of a triangulation on n vertices, i.e. a spanning subgraph of a triangulation.

(b) Every planar graph is an induced subgraph of a triangulation.

(c) Triangulations (on more than 3 vertices) are 3-connected.

(d) Triangulations (on $n \geq 3$ vertices) have exactly $3n - 6$ edges.

(4) Show that an Eulerian plane graph has an Eulerian circuit with no crossings (that is, the curve described by the Eulerian circuit has no proper crossings, however, touching is allowed).

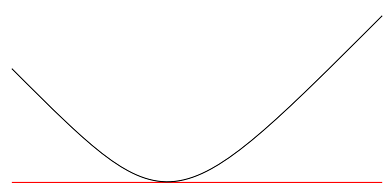

Figure 1: A touching of two curves.

(5) Let G be a graph on the vertex set $v_1, \ldots, v_n$. The graph $M(G)$ is the graph with vertex set $v_1, \ldots, v_n, u_1, \ldots, u_n, w$ such that there is a copy of G on $v_1, \ldots, v_n$, each u_i is adjacent to all the neighbours of v_i in G and w is adjacent to $u_1, \ldots, u_n$.

(a) Show that if G is triangle-free, then $M(G)$ is triangle-free.

(b) Show that if $\chi(G) = k$ then $\chi(M(G)) = k + 1$.

(c) Show that if G is k-color critical[1], then $M(G)$ is $k+1$-color critical.

[1] A graph G is k-color critical if it $\chi(G) = k$ but $\chi(G \setminus e) = k - 1$ for any $e \in E(G)$, and G has no isolated vertices.

(6) [Christmas Bonus Exercise]

(a) Show that for all $k \in \{3, 4, 5\}$ every k-gon is star-shaped. [A set $S \subset \mathbb{R}^n$ is star-shaped if a point $p \in S$ exists, such that for all points $q \in S$, the line segment pq is completely contained in S.]

(c) Show that there is a 6-gon which is not star-shaped.

(c) Deduce from this that every triangulation has a planar drawing, such that all edges are line segments. [Hint: https://www.sfu.ca/~agwesole/Hinweis5]

Frohe Weihnachten
&
Guten Rutsch