(1) Show the following identity:

$$2n^{n-3} = \sum_{k\geq 0} \binom{n-2}{k} (k+1)^{k-1}(n-k-1)^{n-k-3}.$$

[Hint: Count the number of spanning trees of K_n containing a fixed edge.]

(2) Lights Out

(a) Let the cycle with n edges and n vertices be called C_n. For which $n \in \mathbb{N}$ is it always possible to turn all lights of the cycle off, no matter which lights are on and/or off? For all other n, how many combinations of lights that are on and off can be switched off?

(b) Do the above exercise but instead of the cycle C_n, consider the hypercube H_n.

(3) Counting spanning trees

(a) Use the Matrix-Tree Theorem to prove Cayley's formula.

(b) Prove the number of spanning trees of $K_{n,m}$ obtained in exercise 1 from exercise sheet 3 using the Matrix-Tree Theorem.

(4) Let $G = (V, E)$ be a graph and $F \subset E$ a subset of edges. Show the following statements.

(a) You can add edges to F to become a member of the cut space $S(G)$ if and only if F does not contain an odd cycle.

(b) You can add edges to F to become a member of the cycle space $Z(G)$ if and only if F does not contain an odd cut.
[A cut in this context means a member of the cut space, which is what we called an induced cut at some earlier point.]

(5) Let G be an undirected graph and $\vec{G}$ an *orientation* of G, that is, a graph that contains exactly one of (u, v) or (v, u) for each $\{u, v\} \in E(G)$. Let N be the vertex-edge incidence matrix of $\vec{G}$ and let $\hat{N}$ be N without its first row. For any $F \subseteq E$ we will write $\hat{N}(F)$ for the matrix with the columns of $\hat{N}$ that correspond to F. Let $F \subset E$ with $|F| = n-1$. When does $\left(\det\left(\hat{N}(F)\right)\right)^2 = 1$ hold? What else could it be?
[Remark: This can be used to prove the Matrix-Tree theorem for undirected graphs.]