(1) The definition of transversal Matroids in the lecture was as follows: Let X be a finite set and $T_1, \dots, T_\ell \subseteq X$. We say that $A \subseteq X$ is *independent* iff there is $\pi : A \to \{1, \dots, \ell\}$ such that π is injective and $a \in T_{\pi(a)}$ for all $a \in A$. Let $\mathcal{I}$ be the set of all independent subsets $A \subseteq X$. Prove that $(X, \mathcal{I})$ is a matroid. Hint: Make use of the bipartite graph with bipartition sets X and $\{T_1, \dots, T_\ell\}$ and edges (a, T_i) iff $a \in T_i$. Note that π corresponds to a matching in the bipartite graph.

(2)

(a) Let $M = (X, \mathcal{I})$ be a linear matroid with a basis of size n and a realization in K^d for some field K. Prove $n \leq d$ and construct a realisation of M in K^n.

(b) Prove that the Fano-Matroid has no realization in K^d for all d and all fields K with $\mathrm{char}(K) \neq 2$. Hint: Choose a coordinate system and use determinants, analogously to the proof regarding the Pappus configuration in the lecture and make use of 0-entries. Furthermore use, that $a = -a$ for all $a \in K$ holds exactly in fields K with $\mathrm{char}(K) = 2$.

(3) Let $M = (X, \mathcal{I})$ be a matroid. What is $(M_{\setminus x})^*$ and $(M_{/x})^*$.

(4) Let X be a finite set and $\langle . \rangle : \mathrm{Pot}(X) \to \mathrm{Pot}(X)$ such that

- for all $T \subseteq X$ we have $T \subseteq \langle T \rangle = \langle \langle T \rangle \rangle$,
- for all $T, U \subseteq X$ with $U \subseteq \langle T \rangle$ we have $\langle U \rangle \subseteq \langle T \rangle$, and
- for all $T \subseteq X$, $t \in X \setminus T$ and $s \in \langle T \cup \{t\} \rangle \setminus \langle T \rangle$ we have $t \in \langle T \cup \{s\} \rangle$.

Now let $\mathcal{I}$ be the set of all $A \subseteq X$ such that for all $x \in A$ we have $\langle A \setminus \{x\} \rangle \neq \langle A \rangle$. Prove that $(X, \mathcal{I})$ is a matroid and that the closure operator, as defined in the lecture, has the properties required above.

(5) Let $(X, \mathcal{I})$ be an independence system and not a matroid. Prove that there is a weight function $\omega : X \to \mathbb{R}_{\geq 0}$ such that the greedy algorithm does not work correct on $(X, \mathcal{I})$ with respect to ω.