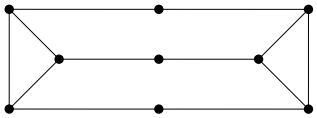
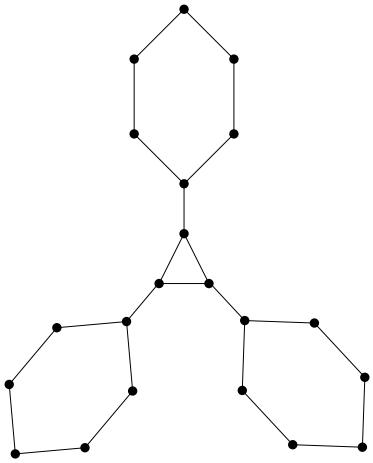
12. Practice sheet for the lecture: **Felsner, Heldt**
Vorlesung über Graphentheorie/ Graphtheory (DS II) 12. January
http://page.math.tu-berlin.de/~felsner/Lehre/dsII11.html

(1) Find the smallest imperfect graph G with $\chi(G) = \omega(G)$ and prove that there is no smaller one.

(2) Are the two graphs below perfect? Do they have perfect orderings?





(3) Let $G = (V, E)$ be a perfect graph. The graph G_v^n is obtained from G by replacing $v \in V$ with a K_n and connecting every vertex $w \in N(v)$ with every vertex of the new K_n; i.e. the vertices of G_v^n are $(V \setminus \{v\}) \cup \{v_1, \ldots, v_n\}$ and the edge set is:

$$(E \setminus \{e \in E \mid v \in e\}) \cup \{\{v_i, v_j\} \mid i, j \in [n] \text{ and } i \neq j\} \cup \{\{v_i, w\} \mid \{v, w\} \in E, i \in [n]\}$$

Prove that G_v^n is perfect. Give one proof, based on the weak perfect graph theorem and one, which uses the strong perfect graph theorem.

(4) A *split graph* is a graph G such that its vertices V can be partitioned into two sets A and B, the graph induced by A is a clique, and the graph induced by B is an independent set.

(a) Prove that a split graph G and its complement $\overline{G}$ are chordal.

(b) Prove that a graph G, which has the property, that G and $\overline{G}$ are chordal does not contain an induced C_4, C_5 or $2K_2$, where $2K_2$ is the graph on 4 vertices, which has two disjoint edges (i.e. the sum of two 2-cliques).

(c) Prove that a graph G, which does not contain an induced C_4, C_5 or $2K_2$ is a split graph.

(d) Prove the the family of split graphs is hereditary (w.r.t. induced subgraphs) and that split graphs are perfect.

(5) *Please hand in your solution of this exercise:*

(a) Let G be an interval graph with n vertices. Prove that G contains at most n inclusion maximal cliques.

(b) Characterize all graphs, which are trees and interval graphs.