**10. Practice sheet for the lecture:** **Felsner, Heldt**
**Vorlesung über Graphentheorie/ Graphtheory (DS II)** 16. December
`http://page.math.tu-berlin.de/~felsner/Lehre/dsII11.html`

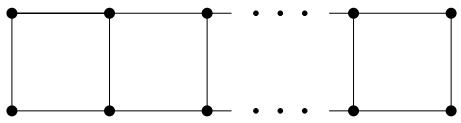
(1) Let $G$ be a bipartite graph. From the lecture we know that $G$ and $\mathcal{L}(G)$ are perfect.

(a) Prove: The complement $\overline{G}$ of $G$ is perfect.

(b) Prove: $\overline{\mathcal{L}(G)}$ is perfect.

(2)

(a) What is the list chromatic number of the graph below?



(b) Let $G$ be a connected graph and $L$ a set of lists of colors for every vertex, such that for every vertex $v$ we have a list of colors $L(v)$ with $|L(v)| \geq \deg(v)$ and there is at least one vertex $w$ such that $|L(w)| > \deg(w)$. Prove that $G$ can be colored properly with the lists in $L$.

(c) Prove or disprove: The list chromatic number of $G \,\square\, H$ equals the maximum of the list chromatic number of $H$ and of $G$.

(3) Let $G$ be a graph and $p_G(x)$ the chromatic polynomial of $G$. Further, Let $G_{/e}$ be the graph, which is obtained by contracting the edge $e \in E$, i.e. merging the endpoints of the edge $e$ and removing all created loops and $G - e$ the graph, which is obtained by deleting $e$. Prove the deletion-contraction formula for $e \in E$:

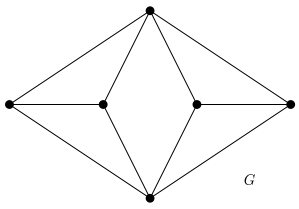
$$p_G(x) = p_{(G-e)}(x) - p_{(G_{/e})}(x)$$

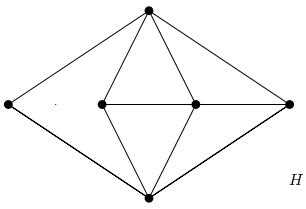
(4) A *quasi-kernel* of a digraph $D$ is a subset of its vertices $U$, which forms an independent set and has the additional property, that for every vertex $v \notin U$ there is a directed path of length $\leq 2$ starting in $v$ and ending in $U$. Show, that every digraph has a quasi-kernel.

Hint: Take a permutation $\pi$ of the vertices of $D$ and split up the edge set of $D$ into *forward* and *backward* edges (i.e. edges $(i,j)$ with $\pi_i < \pi_j$ are forward edges, all other backward edges) and consider the graphs $D_{\rightarrow} := (V(D), \{\text{all forward edges}\})$ and $D_{\leftarrow} := (V(D), \{\text{all backward edges}\})$.

(5) A graph is *chromatically unique* if it is determined by its chromatic polynomial, up to isomorphisms; i.e. there is no non-isomorphic graph, which has the same chromatic polynomial.

(a) Prove that $G$ and $H$ have the same chromatic polynomial.




(b) Prove, that two graphs $G$ and $H$, having the same chromatic polynomial, have the same number of vertices and edges.

(c) Prove that $K_n$ is chromatically unique. Compute the chromatic polynomial of $C_n$.