10. Practice sheet for the lecture: **Felsner, Heldt**
Vorlesung über Graphentheorie/ Graphtheory (DS II) 16. December
`http://page.math.tu-berlin.de/~felsner/Lehre/dsII11.html`

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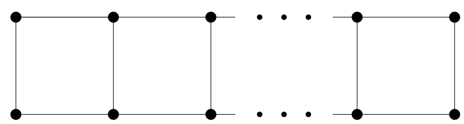
(1) Let G be a bipartite graph. From the lecture we know that G and $\mathcal{L}(G)$ are perfect.

(a) Prove: The complement $\overline{G}$ of G is perfect.

(b) Prove: $\overline{\mathcal{L}(G)}$ is perfect.

(2)

(a) What is the list chromatic number of the graph below?



(b) Let G be a connected graph and L a set of lists of colors for every vertex, such that for every vertex v we have a list of colors $L(v)$ with $|L(v)| \geq \deg(v)$ and there is at least one vertex w such that $|L(w)| > \deg(w)$. Prove that G can be colored properly with the lists in L.

(c) Prove or disprove: The list chromatic number of $G \,\square\, H$ equals the maximum of the list chromatic number of H and of G.

(3) Let G be a graph and $p_G(x)$ the chromatic polynomial of G. Further, Let $G_{/e}$ be the graph, which is obtained by contracting the edge $e \in E$, i.e. merging the endpoints of the edge e and removing all created loops and $G - e$ the graph, which is obtained by deleting e. Prove the deletion-contraction formula for $e \in E$:

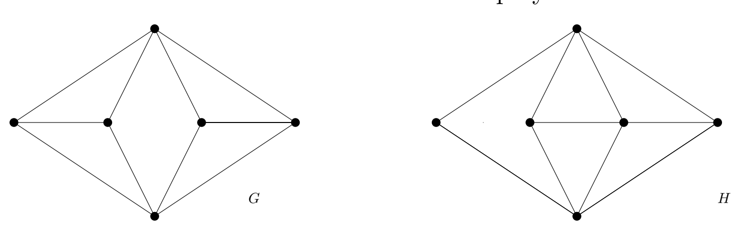
$$p_G(x) = p_{(G-e)}(x) - p_{(G_{/e})}(x)$$

(4) A *quasi-kernel* of a digraph D is a subset of its vertices U, which forms an independent set and has the additional property, that for every vertex $v \notin U$ there is a directed path of length ≤ 2 starting in v and ending in U. Show, that every digraph has a quasi-kernel.

Hint: Take a permutation π of the vertices of D and split up the edge set of D into *forward* and *backward* edges (i.e. edges (i,j) with $\pi_i < \pi_j$ are forward edges, all other backward edges) and consider the graphs $D_{\rightarrow} := (V(D), \{\text{all forward edges}\})$ and $D_{\leftarrow} := (V(D), \{\text{all backward edges}\})$.

(5) A graph is *chromatically unique* if it is determined by its chromatic polynomial, up to isomorphisms; i.e. there is no non-isomorphic graph, which has the same chromatic polynomial.

(a) Prove that G and H have the same chromatic polynomial.



(b) Prove, that two graphs G and H, having the same chromatic polynomial, have the same number of vertices and edges.

(c) Prove that K_n is chromatically unique. Compute the chromatic polynomial of C_n.