7. Practice sheet for the lecture: **Felsner, Heldt**
Vorlesung über Graphentheorie/ Graphtheory (DS II) 23. November
`http://page.math.tu-berlin.de/~felsner/Lehre/dsII11.html`

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(1)

(a) Find $k, n \in \mathbb{N}$ such that $T_k(n)$ is the smallest (with respect to the number of edges) Turán graph, which contains the Petersen graph as subgraph.

(b) Find $k, n \in \mathbb{N}$ such that $T_k(n)$ is the smallest regular Turán graph, which contains the Petersen graph as subgraph.

(2) Let $G = (V, E)$ be a graph with $n \geq 3$ vertices and at least $\left(\frac{1}{4} + c\right) \cdot n^2$ edges for some $c > 0$. Let $d_G := \frac{1}{n} \sum v \in V \deg(v)$ be the average degree of G. Then

$$\begin{aligned}
\#\text{ triangles in } G &\geq \frac{1}{3} \sum_{\{u,v\} \in E} (\deg(u) + \deg(v) - n) \\
&= \frac{1}{3} \left(\sum_{u \in V} \deg(u)^2 - n|E| \right) \\
&\geq \frac{1}{3} \left(n \cdot {d_G}^2 - n|E| \right) \\
&= \frac{1}{3} \left(n \cdot d_G \cdot \left(d_G - \frac{n}{2} \right) \right) \\
&= \frac{1}{3} n \cdot \left(\frac{1}{2} n + 2c \cdot n \right) \cdot (2 \cdot c \cdot n) \\
&\geq 2c \binom{n}{3}
\end{aligned}$$

Explain all steps of this proof in detail.

(3) Let $G = (V, E)$ be a graph with vertices $V = [2n] = \{1, 2, \ldots, 2n\}$ and $V_{\text{blue}} \cup V_{\text{red}} = V$ a partition of its vertices. Further, let $S \subseteq E$ be the cut $[V_{blue}, V_{red}]$, i.e. the set of edges, which have one endpoint in V_{blue} and one in V_{red}. We call S a *balanced cut*, if V_{blue} and V_{red} have the same size. Let $\Omega = \binom{[2n]}{n}$ be the set of balanced cuts of G and $Pr(\omega) = Pr(\omega')$ for all $\omega, \omega' \in \Omega$. What is the expected value of the size of a balanced cut?
Hint: make use of the linearity of the expected value and compute the probability that a fixed edge is in S.

(4) Prove that a simple graph is a complete multipartite graph if and only if it has no 3-vertex induduced subgraph with one edge.

(5) *Please hand in your solution of this exercise:* Let G be a graph on n vertices with average degree d_G. Prove $\alpha(G) \geq \frac{n}{d_G + 1}$.