4. Practice sheet for the lecture: **Felsner, Heldt**

Vorlesung über Graphentheorie/ Graphtheory (DS II) 2. November

`http://page.math.tu-berlin.de/~felsner/Lehre/dsII11.html`

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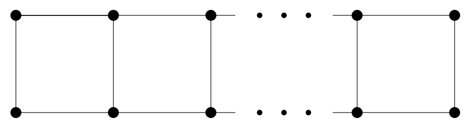
(1) Let $T(n)$ be the number of spanning trees of the graph, which has $2n$ vertices and is shown below.



Prove $T(n) = 4 \cdot T(n-1) - T(n-2)$ for all $n \geq 2$. How fast does $T(n)$ grow for $n \to \infty$?

(2) Let $G = (V, E)$ be a simple graph. The *line graph* $\mathcal{L}(G)$ has vertex set E and two vertices $e, e' \in E$ of $\mathcal{L}(G)$ are connected if and only if they share an endpoint, i.e. if $e \cap e' \neq \emptyset$. A *Hamilton cycle* of a graph G is a cycle, visiting every vertex exactly once. Specify an algorithm which decides if $\mathcal{L}(G)$ has a Hamilton cycle in $\mathcal{O}(|V| + |E|)$ steps, i.e. the algorithm does only a number of steps which is (at most) linear in $|V| + |E|$.

(3)

(a) How many spanning trees does $K_{2,l}$ have? How many classes of isomorphic spanning trees does it have?

(b) How many spanning trees and classes of isomorphic spanning trees does $K_{3,l}$ have?

(c) Let $m \leq n$. How many spanning trees does $K_{m,n}$ have?

(4) Let G be a graph and v a vector in the cycle space of G. Let M be the set of edges of G, whose corresponding entries in v are equal to 1. Characterize M.

(5) *Please hand in your solution of this exercise:* A matroid $M = (E, \mathcal{U})$ consist of a set E and a family $\mathcal{U} \subseteq Pot(E), \mathcal{U} \neq \emptyset$ of subsets of E, which fullfill the following properties:

- $\emptyset \in \mathcal{U}$,
- $A \subseteq B$ and $B \in \mathcal{U}$ imply $A \in \mathcal{U}$, and
- for all $A, B \in \mathcal{U}$ with $|A| < |B|$ exists $x \in B$, such that $A \cup \{x\} \in \mathcal{U}$ holds.

Let $G = (V, E)$ be a graph and $U \subseteq \text{Pot}(E)$ be the set of cycle free sets of edges of G. Prove that (E, U) is a matroid.