3. Practice sheet for the lecture: **Felsner, Heldt**
Vorlesung über Graphentheorie/ Graphtheory (DS II) 27. Oktober
http://page.math.tu-berlin.de/~felsner/Lehre/dsII11.html

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(1)

(a) Determine the switch graph of the degree-sequence $(3, 3, 3, 2, 1)$.

(b) Consider the degree sequence $d = (1, 1, 1, 1, 1, 1)$ and let G be the switch graph of d. How many vertices does G have? What is the degree-sequence of G? Are C_3 and / or C_4 induced subgraphs of G? ...

(2)

(a) Let G be a 3-regular graph. Prove $\kappa(G) = \kappa'(G)$.

(b) Let G be a 4-regular graph. Prove $\kappa'(G) - \kappa(G) \leq 2$.

(c) Let $\mathcal{Q}_d$ be the d dimensional hyper cube. Prove $\kappa(\mathcal{Q}_d) = \kappa'(\mathcal{Q}_d) = d$.

(3) Let G be a graph and a, b distinct and non-adjacent vertices of G. Further, let X an Y be (a, b)-separators, i.e. a and b are in different maximal connected components of $G - X$ and $G - Y$. Let $X_a \subseteq G$ be the set of vertices in G, which are connected to a in $G - X$. Define X_b, Y_a and Y_b accordingly. Consider

$$Z_a := (X \cap Y_a) \cup (X \cap Y) \cup (Y \cap X_a)$$

and

$$Z_b := (X \cap Y_b) \cup (X \cap Y) \cup (Y \cap X_b)$$

and show that Z_a and Z_b separate a and b. Are Z_a and Z_b minimum separators if X and Y are minimum ones? Are Z_a and Z_b minimal separators if X and Y are?

(4) Prove the local directed vertex version of Menger's theorem: Let G be a directed graph and x, y distinct, non-adjacent vertices of G. Then the minimal size of a directed separator equals the maximal number of directed, vertex disjoint xy - paths in G (A directed separator separates x from y, i.e. there is no path from x to y, but not necessarily y from x).

(5) *Please hand in your solution of this exercise:* Prove the global undirected edge version of Menger's theorem: Let G be a finite undirected graph and x, y distinct vertices of G. Then $\kappa'(G)$ equals the minimum over all vertices $x \neq y$ over maximum number of pairwise edge-disjoint paths from x to y.

Hint: For $G = (V, E)$ consider the Graph $G^\star = (V^\star, E^\star)$ with vertex set $V^\star = V \cup E$ and edge set $E^\star = \{(v, e) \in V \times E \mid v \in e\} \cup \left\{(e, e') \in \binom{E}{2} \mid e \cap e' \neq \emptyset\right\}$. To generate $G^\star$ from G put a vertex on every edge and add additional edges in $G^\star$ between any two new vertices, if the corresponding edges in G share an endpoint. Now deduce the undirected edge version of Menger for G from the undirected vertex version of Menger for $G^\star$.