Übungsblatt "Graphs, Order, and Geometry" SoSe 2024

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(1) Show that every nonconstant function on the vertices of a connected graph has at least two poles.

(2) Prove the following: For a connected simple graph G, a nonempty set S of vertices of G and function $h_0 : S \to \mathbb{R}$, there is a unique function $h : V \to \mathbb{R}$ extending h_0 that is harmonic at each vertex of $V \setminus S$.

(3) Consider the random walk on a connected graph G and let S be a nonempty set of vertices of G and function $h_0 : S \to \mathbb{R}$. For vertex v let a_v be the (random) vertex where a random walk starting in v first hits S. Show that the function $h(v) = \mathbf{E}(h_0(a_v))$ is harmonic and extends h_0.

(4) Let G be a 2-connected graph with vertices s and t. Show that there exists a (s,t)-*orientation* of G, i.e., an acyclic orientation such that s is the unique source and t the unique sink of the orientation.

(5) Construct a planar graph G which has no representation by touching squares, i.e, the vertices are represented by squares in the plane so that the interiors of the squares are disjoint and two squares share a boundary point if and only if the corresponding vertices are adjacent.

(6) Let X be the set of points in the plane such that every triple of points of X can be covered by a triangle of area 1. Prove that the convex hull of X can be covered by a triangle of area 4.

(7) (Kirchberger's Theorem) Let X be the set of points in the plane in general position. Suppose points in X are colored with red and blue such that for every four points there is a line separating points of different color. Prove that there exists a line separating the red and the blue points of X.

(8) Let Q be a union of axis parallel rectangles. Suppose for every two points $x, y \in Q$ there exists a point v, such that segments xv and yv are contained in Q. Prove that Q is a star-shaped polygon.