Übungsblatt " Graphs, Order, and Geometry" SoSe 2024

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(1) Prove that the smallest size of a universe required to represent any n-vertex graph ($n \geqslant 4$) as an intersection graph of family of sets from this universe is $\lfloor \frac{n^2}{4} \rfloor$. For a lower bound you should construct a n-vertex graph requiring such a large universe. Hint for an upper bound: try induction on n and use the fact that $\lfloor \frac{(n+2)^2}{4} \rfloor = \lfloor \frac{n^2}{4} \rfloor + n + 1$.

(2) Let $\lambda_3(n)$ be the maximal length of a $(n, 3)$ Davenport–Schinzel sequence, i.e., the maximal length of a sequence over an alphabet with n letters such that the sequence has no consecutive identical letters and no subsequence of type $\ldots a \ldots b \ldots a \ldots b \ldots a \ldots$. Prove the recurrence

$$\lambda_3(n) \;\leqslant\; \lambda_3(n-1) + \frac{\lambda_3(n)}{n} + 2$$

and the implication

$$\frac{\lambda_3(n)}{n} \;\leqslant\; \frac{\lambda_3(n-1)}{n-1} + \frac{2}{n-1} \;\leqslant\; 2\,(1 + \log_2(n)).$$

(3) Let $G = (V, E)$ be a graph with $\chi(G) > a \cdot b$ and let $G_1 = (V, E_1)$ and $G = (V, E_2)$ be subgraphs of G with $E_1 \cup E_2 = E$. Show that $\chi(G_1) > a$ or $\chi(G_2) > b$.

(4) A *shift graph* of size n is a graph H_n such that $V(H) = \{(i, j) \mid 1 \leqslant i < j \leqslant n\}$ and (i, j), (k, l) are adjacent iff $i < j = k < l$. Prove that $\chi(H_n) = \Theta(\log n)$. Thus, give an argument for a lower and an upper bound. Hint for a lower bound: for any fixed coloring of H_n and every $1 \leqslant i \leqslant n$ consider the set C_i of colors used for vertices of the form $\{(i, j) \mid i < j\}$. Prove that $C_i \neq C_{i'}$ for $i \neq i'$.

(5) Construct a graph which is *not*

a) an intersection graph of axis-aligned rectangles;

b) an intersection graph of segments.

(6) Mimicking the construction of triangle-free segment graphs with high chromatic number (Pawlik et al.) show that for every $n \geqslant 1$ there is a triangle-free family $\mathcal{F}_n$ of L-shapes (shapes consisting of a horizontal and a vertical segments sharing an endpoint) such that $\chi(\mathcal{F}_n) > n$.

(7) Show that $\mathcal{S}_n$, the triangle-free family of segments with high chromatic number constructed at the lecture, has an independent set of size $\frac{1}{4}|\mathcal{S}_n|$.

(8) The abstract graphs represented by the family $\mathcal{S}_n$ of segments are known as *Burling graphs*. Describe a recursive construction of Burling graphs.