Technische Universität Berlin Summer 2025

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Combinatorics (DS I) - Sheet 11

Exercise 11.1.

Consider necklaces with 12 beads of at most three different colors.

(a) How many different necklaces exist up to rotations of the necklace?

(b) How many different necklaces exist up to rotations and mirrorings of the necklace?

Exercise 11.2.

Consider two magicians Alice and Bob in well separated rooms. A volunteer picks five cards from a standard deck (52 cards) and hands them to Alice. Alice keeps one of the five cards and puts the other four (in specific order) in an envelope. The envelope is brought to Bob who opens it, has a look at the cards and announces the fifth card.

(a) Explain the existence of a strategy for this trick with the aid of Hall's Theorem.

(b) Find a playable strategy.

Exercise 11.3.

Let $(\mathcal{P}, \mathcal{B})$ be a $S_\lambda(t, k, v)$ design.

(a) Let $p \in \mathcal{P}$ and $\mathcal{B}_{-p} = \{B \in \mathcal{B} \mid p \notin B\}$ be the set of blocks which do not contain p. Show that $(P \setminus \{p\}, \mathcal{B}_{-p})$ is a design and determine its parameters.

(b) Consider the complement of a $S_\lambda(t, k, v)$ design, i.e. replace each block by its complement. Prove that the complement of a $S_\lambda(t, k, v)$ design is a design for the same parameter t. Determine its other parameters.

Exercise 11.4.

Consider the tetrahedron. Compute its cycle index polynomial under rigid transformations and use this to compute the number of different edge colorings of it with 4 colors.

Exercise 11.5.

A design $(\mathcal{P}, \mathcal{B})$ is resolvable, if there is a partition of $\mathcal{B}$ into partitions of $\mathcal{P}$. Let $(\mathcal{P}, \mathcal{B}) = S(2, n+1, n^2 + n + 1)$ be a projective plane and fix $B \in \mathcal{B}$. Show that

$$\Big(\mathcal{P} \setminus B, \{C \setminus B \mid C \in \mathcal{B} \setminus \{B\}\}\Big)$$

is a resolvable $S(2, n, n^2)$ design.

Bonus Exercise

Let there be 27 cubes of volume 1. Count the number of ways to paint the faces of the cubes with 3 colors, such that for any color c, you can stick the cubes together to form a cube of volume 27 whose faces are of color c, up to dihedral symmetries of the 27 cubes it consists of and their order.

List of hints

- Use Pólya theory, but not for the cubes, there you need a system of linear equations.
- The configurations of rooks correspond to permutations.