Technische Universität Berlin Summer 2025
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Combinatorics (DS I) - Sheet 10

Exercise 10.1.

(a) Given a symmetric chain C in the Boolean lattice $\mathcal{B}_n$, is there always a symmetric chain decomposition containing C?

(b) Show that the number of chains of length $n+1-2k$ in a symmetric chain decomposition of $\mathcal{B}_n$ is $\binom{n}{k} - \binom{n}{k-1}$.

(c) Use (b) to derive the known explicit formula for the Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$.

Exercise 10.2.

Let $\mathcal{B}_n^\vee$ be the truncation of the Boolean lattice where the maximal and minimal element is deleted. Let $\mathcal{C}$ be a symmetric chain decomposition which is canonical (originating from the bracketing process). Let $\bar{\mathcal{C}}$ be its complement, i.e. for a chain $C \in \mathcal{C}$ the set $\bar{C}$ of complements of sets in C is a set in $\bar{\mathcal{C}}$.

(a) Show that $\bar{\mathcal{C}}$ is a symmetric chain decomposition.

(b) Show that $\mathcal{C}$ and $\bar{\mathcal{C}}$ are orthogonal, i.e. $|C \cap D| \leq 1$ for all $C \in \mathcal{C}$ and $D \in \bar{\mathcal{C}}$.

Exercise 10.3.

Prove or disprove: Let $P = (X, \leq)$ be a poset and C a maximum chain in it. Then the width of $P' = (X \backslash C, \leq)$ is smaller than the width of P. What if C is a maximal chain? Can this be used to prove Dilworth's theorem?

Exercise 10.4.

(a) What is the number C_n^k of chains of size k in the Boolean lattice $\mathcal{B}_n$?

(b) Prove by bijection, that the number of antichains of size 2 is C_n^3.

(c) What is the number of antichains of size 3 in $\mathcal{B}_n$?

Exercise 10.5.

Find an infinite counterexample to the Theorem of Hall, i.e., find a bipartite graph $G = (X \cup Y, E)$ with the property that $|N(S)| \geq |S|$ for all $S \subset X$ and there is no matching containing all vertices of X.

Exercise 10.6.

In the lecture it was proven that if poset P has an orhogonal chain decomposition C_1, C_2 with $|C_1| = k$, $|C_2| = \ell$, then two randomly chosen elements $x, y \in P$ (with any probability distribution) fulfill, then the probability of $x \leq y$ is at least

$$\frac{1}{2}\left(\frac{1}{k} + \frac{1}{\ell}\right).$$

(a) Look through the proof and identify the conditions that need to be met for equality to hold for all pairs $x, y \in P$.

(b) Find an example where equality holds if k divides ℓ.

Bonus Exercise

Consider the infinite Boolean lattice, i.e. the set of all subsets of $\mathbb{N}$ with subset relation. Show that the height of this poset is $|\mathbb{R}|$ by construction an uncountable chain.

List of hints

- The complement is obtained by interpreting the brackets differently.
- Valid bracket expressions are counted by the Catalan numbers.
- In a chain of size two, what happens two times?
- There is a vertex connected to all on the other side.
- $\mathbb{Q}$.