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Combinatorics (DS I) - Sheet 9

Exercise 9.1.

Let $\mathcal{A}$ be a family of k-subsets of $[n]$ such that for every h-tuple $(A_1, \ldots, A_h)$ of sets from $\mathcal{A}$ we have $A_1 \cap \ldots \cap A_h \neq \emptyset$. Show that if $k \cdot h \leq (h-1)n$, then

$$|\mathcal{A}| \leq \binom{n-1}{k-1}.$$

Exercise 9.2.

Let $F_k(m)$ be the collection of the first m sets in the colex order on k-sets. Show that for $m \geq 1$ we have

$$|\triangle F_k(m+1)| \leq |\triangle F_k(m)| + (k-1).$$

Exercise 9.3.

Let $\mathcal{A}$ be a family of k-subsets of $[n]$ and $\mathcal{B}$ a family of ℓ-subsets of $[n]$ such that $\ell + k \leq n$ and $A \cap B \neq \emptyset$ for every $A \in \mathcal{A}, B \in \mathcal{B}$.

(a) Show that either

$$|\mathcal{A}| \leq \binom{n-1}{k-1},$$

or

$$|\mathcal{B}| \leq \binom{n-1}{\ell-1}.$$

(b) Use this to prove the Erdős-Ko-Rado theorem

Exercise 9.4.

(a) In a town there are $2n$ people and $\binom{x}{k}, x \in \mathbb{R}$ clubs with k people each. In the evening, all people of the town are in one of two bars. Show that there is a set of $k-1$ people who are in the same club and went to the same bar if

$$\binom{x}{k-1} > \binom{2n}{k-1} + 2\binom{n}{k-1}.$$

Bonus Exercise

Given a permutation $\pi = \pi(1), \dots, \pi(n-1)$, we define $i_k^n(\pi) = \pi(1), \dots, \pi(k-1), n, \pi(k), \dots, \pi(n-1)$, for $k = 0, \dots, n-1$ (for $k = 0, n-1$, n is inserted before/after all other elements). Furthermore, we define $\overleftarrow{c}_n(\pi) = i_{n-1}^n(\pi), \dots, i_0^n(\pi)$ and $\overrightarrow{c}_n(\pi) = i_0^n(\pi), \dots, i_{n-1}^n(\pi)$.

The *Steinhaus-Johnson-Trotter Gray code* (SJT(n)) is a cyclic listing of all permutations of $[n]$, such that consecutive permutations differ only in an adjacent transposition. It is defined recursively by setting SJT(1) = 1, and SJT(n) is obtained from SJT($n-1$) by replacing π_{2k-1}, π_{2k} by $\overleftarrow{c}_n(\pi_{2k-1})\overrightarrow{c}_n(\pi_{2k})$, for $k = 1, \dots, \frac{(n-1)!}{2}$.

(a) Show that SJT(n) really is a cyclic listing of all permutations of $[n]$ such that consecutive permutations differ in an adjacent transposition.

(b) Given $p \in [n!]$, how can you determine the permutation in position p of SJT(n) efficiently?

(c) How can you define SJT(n) without using recursion?

List of hints

- Consider the parity of the right sub-permutation.
- Use shadows as in the second proof of the Erdős-Ko-Rado theorem.
- For each number in the permutation, keep track in which direction it should move next.
- There is a very trivial inequality that is just different by 1.
- Consider the proof of Erdő-Ko-Rado using the cyclic permutations. There is a key point which is not true if $n < 2k$, but it is fixed by the stronger intersection property.