Technische Universität Berlin Summer 2025
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Combinatorics (DS I) - Sheet 8

Exercise 8.1.

Let $P = (M, \leq)$ be a finite poset with n elements. For a given ordering of the elements $(m_1, \ldots, m_n)$, we can associate with P a relation matrix $(a_{ij}) = A$, such that $a_{ij} = 1 \iff m_i \leq m_j$.

(a) Show that it is possible to find an ordering $(m_1, \ldots, m_n)$ such that A is an upper triangle matrix.

(b) Show that
$$|\{(i,j) \mid a_{ij} = 1\}| = \binom{n+1}{2} \iff P \text{ is a total order.}$$

(c) Find matrix properties for a $\{0,1\}$-Matrix B which ensure that B represents a partial order relation.

Exercise 8.2.

A poset $P = (X, \leq)$ is *ranked*, if there exists a function $r \colon X \to \mathbb{N}$ such that $r(x)$ is the length of *all* maximal chains ending in x. For $n \in \mathbb{N}$, the *divisor-poset* P_n is the set of all divisors of n ordered by divisibility:
$$P_n = (\{x \mid x \text{ divides } n\}, \{(x,y) \mid x \text{ divides } y \text{ and } y \text{ divides } n\}).$$

Prove that P_n is ranked. Sketch the Hasse-diagram of P_{60} and compute its ranks. Is P_n a lattice?

Exercise 8.3.

(a) Give a simple combinatorial proof of the Erdős-Ko-Rado theorem for the case $n = 2k$. How many different maximal intersecting families exist?

(b) Let $n \geq 2k$, and let $\mathcal{F}$ be a family of k-element subsets of $[n]$ with the following properties:

- No collection of r sets $A_1, \ldots, A_r$ in $\mathcal{F}$ is pairwise disjoint, i.e. there always is a pair A_i, A_j with $A_i \cap A_j \neq \emptyset$, $i \neq j$.
- If A and B intersect some set C, then A and B intersect each other (i.e. the intersecting relation is transitive on $\mathcal{F}$).

Prove that
$$|\mathcal{F}| \leq (r-1)\binom{n-1}{k-1}.$$

Exercise 8.4.

(a) How many different maximal chains of nested subspaces of $V_n(q)$ exist?

(b) Define the $q-$ analog of the boolean lattice. Consider an antichain $\mathcal{A}$ and show the $q-$analogue of the LYM inequality:

$$\sum_{k=0}^{n} \frac{p_k(\mathcal{A})}{\begin{bmatrix} n \\ k \end{bmatrix}} \leq 1.$$

Exercise 8.5.

(a) Show that every finite poset is isomorphic to the containment order on some family of sets.

(b) Show that every finite poset can be obtained by taking a subset of integers with the divisor relation.

(c) Find a poset with an element that is greater than every other and an element that is smaller than any other, but which is not a lattice.

Bonus Exercise

Prove the following identity:

$$\sum_{k=0}^{n} \binom{2k}{k}\binom{2n-2k}{n-k} = 4^n.$$

List of hints

- For the bonus: consider gridpaths and let the weight of a grid path be the number of times it touches the diagonal.
- Consider the set of principal ideals.
- The $q-$analogue from (b) has something to do with (a).
- Mirror the path across the right diagonal.
- Look at the proof from the lecture and translate into the q setting.
- Prime numbers.
- Split the path when it touches the diagonal.
- The intersecting families are very structured in this case.