Technische Universität Berlin Summer 2025
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Combinatorics (DS I) - Sheet 7

Exercise 7.1.

Use two different interpretations of the q-binomials to show the following equations:

(a)

$$\sum_{i=0}^{n} \begin{bmatrix} i \\ k \end{bmatrix} q^{(k+1)(n-i)} = \begin{bmatrix} n+1 \\ k+1 \end{bmatrix},$$

(b)

$$\begin{bmatrix} n \\ m \end{bmatrix} \begin{bmatrix} m \\ k \end{bmatrix} = \begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} n-k \\ m-k \end{bmatrix}.$$

Exercise 7.2.

Prove the following about q-binomials:

(a)

$$\prod_{k=1}^{n} (x + q^k y) = \sum_{k=0}^{n} q^{\binom{k+1}{2}} \begin{bmatrix} n \\ k \end{bmatrix} y^k x^{n-k},$$

(b)

$$\sum_{k \geq 0} \binom{n+k}{k} z^k = \prod_{i=0}^{n} \frac{1}{1 - q^i z}.$$

Exercise 7.3.

Search the literature for at least three different combinatorial structures which are counted by the Catalan numbers, at least one of which is from a book and not on Wikipedia. Give proofs that they really are counted by the Catalan numbers. You are allowed to present a proof you found in the literature.

Exercise 7.4.

Let $\mathcal{B}$ be a countable set with a weight function $|\cdot|\colon \mathcal{B} \to \mathbb{N}$ and let $\mathcal{P}_f(\mathcal{B})$ be the set of finite subsets of $\mathcal{B}$. Show that the generating function A of $\mathcal{P}_f(\mathcal{B})$ can be expressed as

$$A(x) = \prod_{n \geq 0} (1 + z^n)^{b_n},$$

where $A(x) = \sum_{a \in \mathcal{P}_f(\mathcal{B})} x^{|a|}$ and $B(x) = \sum_{b \in B} x^{|b|} = \sum_n b_n x^n$ is the generating function of $\mathcal{B}$. Think about the right extension of the weight function to finite subsets for this to make sense.

Exercise 7.5.

We consider the binary Pascals-Triangle, which consists of the entries $\binom{n}{k}$ mod 2.

(a) Draw the triangle up to $n = 16$ and formulate a conjecture for how it continues.

(b) Prove your conjecture.

(c) Research the statement of Lucas' Theorem. Can you give a proof sketch of this theorem using the ideas used to prove (b)?

Bonus Exercise

Alice writes all lists of numbers $a_1, \dots, a_n \in \mathbb{N}$ (including 0) such that $\sum a_i \leq k$ and Bob writes all lists of numbers $b_1, \dots, b_k \in \mathbb{N}$ such that $\sum b_i \leq n$. Show that they wrote the same number of lists.

Now they write the lists with numbers $a_1, \dots, a_n \in \mathbb{Z}$ and $b_1, \dots, b_k \in \mathbb{Z}$ such that $\sum |a_i| \leq k$ and $\sum |b_i| \leq n$. Did they still write the same number of lists?

List of hints

- In all statements about q-binomials, consider their interpretation, for example as grid paths.
- The weight of a subset if the sum of the weights of its elements.
- If n is a power of two, you get a row of 1's. From there you can find a recursion.
- There is a relation to 1,0 strings with the right number of 0's and 1's.
- Think about the combinatorial proof of the binomial theorem.
- Walk i steps up, then right, and then the rest.