Combinatorics (DS I) - Sheet 5

Exercise 5.1.

Let a_k denote the number of words of length k over the alphabet $\{u, l, r\}$ with no l and r consecutive, i.e. lr and rl do not appear. These words can be interpreted as grid paths of length k which go up, left and right, and do not intersect themselves.

  (a) Find a linear recursion for a_k. **Hint:** If you find a recursion with $n-1$ terms, calculate $a_n - a_{n-1}$.

  (b) Express the generating function $A(z)$ as a rational function.

  (c) Find a closed form for a_k.

Exercise 5.2.

Alice and Bob play a game. They place n coins on the table and take turns removing them. In a turn, either 2,5 or 6 coins are removed. If a player cannot make a move, they lose. Who has a winning strategy if Alice begins?

Exercise 5.3.

In how many ways can you pay n Euro with 1, 5 and 10 Euro notes/coins?

Exercise 5.4.

  (a) Let $g(n)$ be the number of subsets of $[n]$ that contain no two consecutive numbers. Find a recurrence relation for g.

  (b) Use a bijection to prove the following about the Fibonacci numbers:

$$F_{n+2} + \sum_{k=2}^{n} 2^{n-k} F_{k-1} = 2^n.$$

Exercise 5.5.

Prove the following fact about Fibonacci numbers:

  (a)

$$f_{2n-1} = \sum_{k=1}^{n} \binom{n}{k} f_{k-1}.$$

  (b)

$$3f_n = f_{n+2} + f_{n-2}.$$

Bonus Exercise

Alice, Bob, Charlie and you are in a bar. You know that they are a mathematician, a physicist and an engineer, but you forgot who is what. Unfortunately for you, they agreed to mess with you a bit, so they decided to only answer your questions with foo or bar. The mathematician decided on some convention for which word means yes and which means no, so naturally, the physicist decided to use the opposite convention. The engineer had trouble following the plan, so they decided to just answer every question randomly with foo or bar. Because the music in the bar is so loud, you can only ever talk to one of the three at a time. How many questions do you need to ask in total to determine the professions of all three?