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Combinatorics (DS I) - Sheet 4

Exercise 4.1.

(a) How many subsets of the set $[n]$ contain at least one odd integer?

(b) How many multi-subsets[a] of the set $[n]$ have size k?

(c) For $k \in [n]$, how many sequences $(T_1, \ldots, T_k)$ are there with $T_0 = \emptyset \subsetneq T_1 \subsetneq \ldots \subsetneq T_{k-1} \subsetneq [n] = T_k$?

[a] Multi-sets may contain the same element more than once.

Exercise 4.2.

For which $n \in \mathbb{N}$ can you find pairwise different $x_1, \ldots, x_n \in [n+1]$ such that the values $|x_i - x_i + 1|, i = 1, \ldots n-1$ and $|x_1 - x_n|$ are pairwise different?

Exercise 4.3.

Show that there exists a polynomial R, such that:

$$\prod_{k=1}^{s} (1 + x^k + x^{2k} + ... + x^{sk}) = \sum_{n=1}^{s} p(n) x^n + x^{s+1} R(x),$$

where $p(n)$ is the number of partitions of n.

Exercise 4.4.

Let $T(n,k)$ be the number of partitions of $[n]$ such that no two blocks contain consecutive numbers. Show:

$$T(n,k) = S(n-1, k-1),$$

(a) by using recursion.

(b) by using a bijection. **Hint:** The block containing n is special.

Exercise 4.5.

Show

$$S(n+1, m+1) = \sum_k \binom{n}{k} S(k, m).$$

Bonus Exercise

Given n blue points on a circle. How many red points must be added such that there is no triangle with distinct blue points as vertices which does not contain a red point? **Hint:** Draw an edge between any pair of blue points and consider the regions.