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Combinatorics (DS I) - Sheet 3

Exercise 3.1.

Consider the randomized Quick-Sort algorithm. Given a permutation L of n distinct integers, perform these steps:

- Pick, at random, an element $x \in L$.
- Split the elements of $L \backslash \{x\}$ into A and B, such that $a < x < b$ for any choice $a \in A$ and $b \in B$.
- Sort A and B using Quick-Sort.

(a) Define the random variable $X_{i,j}$ as

$$X_{i,j} = \begin{cases} 1, & i \text{ and } j \text{ are compared during the sorting,} \\ 0, & \text{else.} \end{cases}$$

What is the probability that $X_{i,j} = 1$? Hint: Consider the moment at which i and j are separated.

(b) Using a Zauberzeile, what is the expected number of comparisons of randomized Quick-Sort?

Exercise 3.2.

Given some number of points in $\mathbb{Z}^d$, we seek sets of 3 points a, b, c such that their midpoint $\frac{1}{3}(a + b + c)$ also has integer coordinates.

(a) How many points do I need to give you until there is always such a set if $d = 2$?

(b) What is the answer for $d = 3$?

(c) What is the connection to the game SET?

Exercise 3.3.

Given some $n \in \mathbb{N}$, what is the number of permutations of $[n]$ that have a cycle of length $> \frac{n}{2}$?

Exercise 3.4.

In the parliament of some country there are $2n+1$ seats filled by 3 parties. How many possible distributions (i, j, k), (i.e. party 1 has i, party 2 has j, and party 3 has k seats) are there, such that no party has an absolute majority? Give a nice combinatorial proof. (Hint: Look at small numbers and make a good guess.)

Exercise 3.5.

In a weird math prison, $2N$ prisoners are given a chance of release. $2N$ boxes are places from left to right along a wall. The boxes contain cards with the names of the prisoners, one card per box, such that every prisoner is secretly associated to some box. After the prisoners had a chance to agree on a strategy, they are lead to the boxes one by one. Each prisoner is allowed to look into exactly N boxes, but is not allowed to change anything about the setup. Afterwards they are not allowed further communication with any other prisoner. If all prisoners find their own name in the N attempts, all prisoners are granted freedom. If only one prisoner fails to find their name, nobody is granted freedom.

Find a strategy, such that the probability of success converges to $1 - \ln(2)$ for large N.