Technische Universität Berlin Summer 2025
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Combinatorics (DS I) - Sheet 2

Exercise 2.1.

Let $x^{\underline{n}} := x \cdot (x+1) \cdots (x+n-1)$ denote the raising factorials. Deduce the following equation from Vandermonde's identity:

$$(x+y)^{\underline{n}} = \sum_{k=0}^{n} \binom{n}{k} x^{\underline{k}} y^{\underline{n-k}}, \qquad \left[\textbf{Hint: } \binom{-x}{k} = (-1)^k \binom{x+k-1}{k}\right]$$

Exercise 2.2.

Find a closed form expression for A_n:

$$A_n = \sum_{k=0}^{n} \binom{n-k}{k} (-1)^k.$$

Exercise 2.3.

A permutation $\pi \in S_n$ is a transposition if the position of exactly two elements is switched, i.e. n cyclic notation, there are $n-2$ fixed points and a cycle of length 2.

(a) Show that each permutation is a composition of transpositions. We denote the minimum number of transpositions needed to express π by $t(\pi)$.

(b) Let $\pi \in S_n$ be a permutation with exactly one cycle of length k and $n-k$ fixed points. Show that $t(\pi) = k-1$.

Exercise 2.4.

There are nine caves in a forest, each being the home of one animal. Between any two caves there is a path, but they do not intersect. Since it is election time, the candidating animals travel through the forest to advertise themselves. Each candidate visits every cave, starting from their own cave and arriving there at the end of the tour again. Also, each path is only used once in total, since nobody wants to look at the embarrassing posters (not even their own), put there on the first traverse. Under these assumptions, what is the maximum possible number of candidates?

Exercise 2.5.

Mutually orthogonal latin squares (MOLS)

(a) Prove one part of the unproved theorem from the lecture: Construct $(n-1)$ MOLS of order n from a projective plane of order n.

(b) Let F be a field of n elements. For all $q \in F\setminus\{0\}$, define a $n \times n$ table Q_q by $Q_q(x, y) = qx + y$. Show that these tables are MOLS.

Bonus Exercise

A Boolean function $f\colon \binom{[n]}{2} \to \{0, 1\}$ is said to avoid a pattern $\sigma \in \{0, 1\}^3$, if for no triple $a < b < c \in [n]$ we have $(f(a, b), f(a, c), f(b, c)) = \sigma$. Consider the set of functions avoiding all patterns in $\{(1, 1, 0), (1, 0, 1), (0, 1, 1)\}$.

(a) Write some code to compute the number of such functions for $n = 1, \ldots, 7$ or more.

(b) Paste this sequence of numbers into the OEIS to obtain an educated guess at an interpretation of these numbers.

(c) Prove your guess.