Combinatorics (DS I) - Sheet 1

Exercise 1.1.

A spider has a particular sock and a particular shoe for each of his eight feet. In how many different orders can it put on his shoes and socks, assuming that on each foot it has to put on the sock first?

Exercise 1.2.

The squares of a 4×7 chessboard are coloured arbitrarily black and white. Show that there is an $i \times j$-rectangle with $i \geq 2$ and $j \geq 2$, such that all four of its corners have the same colour. Is the same true for a 4×6 chessboard?

Exercise 1.3.

Consider a chess tournament of n players, each playing against every other participant. Show that at each point of time during the tournament there exist at least two players having finished the same number of games.

Exercise 1.4.

A sequence of numbers $a_1, \ldots, a_n$ is called *unimodal*, of there exists an m, such that $a_i \leq a_{i+1}$ for all $i < m$ and $a_i \geq a_{i+1}$ for all $i \geq m$. Give three different proofs of the unimodalitz of the sequence $\binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n}$ for all $n \in \mathbb{N}$ based on the three given hints:

(a) Use the definition

$$\binom{n}{k} := \frac{n!}{k!(n-k)!}.$$

(b) Consider therecursive definition

$$\binom{n}{k} := \binom{n-1}{k-1} + \binom{n-1}{k} \quad \text{and} \quad \binom{n}{0} = \binom{n}{n} = 1.$$

(c) Use the interpretation that $\binom{n}{k}$ is the number of k-element subsets of $[n]$.

Exercise 1.5.

(a) Give an alternative proof that the generating function of derangements is given by

$$D(z) := \sum \frac{d(n)}{n!} z^n = \frac{e^{-z}}{1-z}$$

by manipulating $D(z)e^z$ and using an appropriate identity from the lecture.

(b) Prove the following identity:

$$d(n) = nd(n-1) + (-1)^n.$$

(c) How many permutations in S_n are derangements and involutions, where an involution is a permutation that is its own inverse?

(d) What is the probability that a random involution is a derangement as n approaches infinity? In particular, is this probability well defined?